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Equivalence of a mixing condition and the LSI in spin systems with infinite range interaction

Christopher Henderson[a,*], Georg Menz[b]

[a] *Unité Mathématiques Pures et Appliquées, École Normale Supérieure de Lyon, France*
[b] *Department of Mathematics, UCLA, United States*

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Abstract

We investigate unbounded continuous spin-systems with infinite-range interactions. We develop a new technique for deducing decay of correlations from a uniform Poincaré inequality based on a directional Poincaré inequality, which we derive through an averaging procedure. We show that this decay of correlations is equivalent to the Dobrushin–Shlosman mixing condition. With this, we also state and provide a partial answer to a conjecture regarding the relationship between the relaxation rates of non-ferromagnetic and ferromagnetic systems. Finally, we show that for a symmetric, ferromagnetic system with zero boundary conditions, a weaker decay of correlations can be bootstrapped.

Keywords: Spin systems; Log-Sobolev inequality; Poincaré inequality; Decay of correlations; Phase transition; Gibbs measure; Long-range interactions

1. Introduction

In this article we consider a system of continuous real-valued spins on a subset of the lattice $\mathbb{Z}^d$. The system is described by its finite volume Gibbs measures μ_Λ^ω (for a precise definition see Section 2.1). Here, $\Lambda \subset \mathbb{Z}^d$ denotes the domain on which the Gibbs measure is active, and ω denotes the boundary values. Recently, some progress has been made in transferring classical results for finite-range interaction to infinite-range interaction. For instance, classical results regarding the absence of a phase-transition for systems on a one-dimensional lattice when

* Corresponding author.
E-mail addresses: christopher.henderson@ens-lyon.fr (C. Henderson), gmenz@math.ucla.edu (G. Menz).

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the interaction is finite-range have recently been shown to hold when the interactions decay algebraically of order $2+\alpha$ [44,31].

In that spirit, we generalize in this article another classical result to infinite-range interaction. In higher dimensions, the situation is murkier than the one described above; phase transitions can occur. In other words, the behavior of the system may depend on the size of the lattice and the boundary conditions imposed. In physics, the high temperature region of a system is characterized by a sufficient decay of correlations. We show that the high temperature region, where phase transitions do not occur, can be characterized by a uniform spectral gap or a uniform log-Sobolev inequality. The connection between mixing conditions on the Gibbs measure and the relaxation of the associated Glauber dynamics has a long history dating back to the 1970s [18,39]. Stroock and Zegarlinski [39,40], in the compact spin space setting, showed that when the interactions have finite range, the following three conditions are equivalent:

1. The finite volume Gibbs measures μ_Λ^ω satisfies the Dobrushin–Shlosman mixing condition.
2. The finite volume Gibbs measures μ_Λ^ω satisfy a logarithmic Sobolev inequality (LSI) uniform in the system size $|\Lambda|$ and the boundary condition ω.
3. The finite volume Gibbs measures μ_Λ^ω satisfy a Poincaré inequality (PI) uniform in the system size $|\Lambda|$ and the boundary condition ω.

In the setting of a compact or discrete spin space, this result was generalized in 1995 by Laroche [21] to infinite-range interaction. The case of an unbounded spin space is technically more challenging due to the loss of compactness. Therefore it took until 2001 when Yoshida [43] was able to obtain this type of result to unbounded spins with finite-range interaction. In this article we further generalize this statement to unbounded spins with infinite-range interaction.

We first describe what the conditions (1)–(3) mean. We start by describing condition (1) regarding mixing conditions. There are a variety of different mixing conditions, see for example [7–9] or [24], and many of them are equivalent. Among the mixing conditions, the most famous is the Dobrushin–Shlosman mixing condition (see [7]). The Dobrushin–Shlosman mixing condition referred to above is known to be true for example when the interaction is weak enough. The common thread between all mixing conditions is that each condition quantifies the influence on the finite-volume Gibbs measures μ_Λ^ω of changing a spin value over large distances and therefore are describing a static property of μ_Λ^ω. Mixing conditions are quite useful in the study of spin-systems. As we mentioned above, one can use them to describe the high-temperature regime of the system. In this article, we show that a uniform PI yields the following mixing condition: The covariance of functions of bounded support with respect to the finite-volume Gibbs measures μ_Λ^ω decays algebraically of order $d+\alpha$, $\alpha>0$, in the distance between the supports of each function. We show in Lemma 2.3, that this condition is equivalent to the Dobrushin–Shlosman mixing condition. For closing the loop of equivalences, we use the special case of spin–spin correlations decaying algebraically in the distance of the sites.

We now describe conditions (2) and (3). The PI and the LSI are associated to properties of the Glauber dynamics of the system, the naturally associated diffusion process on the state space of the system. Both yield exponential decay to equilibrium of this system, which is described by the Gibbs measure μ_Λ^ω (for details see for example the introduction in [28]). The LSI was originally introduced by Gross [15] and yields the PI by linearization (see for example [22]). The difference between the LSI and the PI is that the LSI yields exponential convergence with respect to relative entropies, whereas the PI only yields convergence with respect to variance. This difference becomes important when looking at high-dimensional situations or continuum limits i.e. sending $|\Lambda|\to\infty$ (see for example [16,12]). Here, the LSI is used to derive quantitative