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Volatility of Boolean functions

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Abstract

We study the volatility of the output of a Boolean function when the input bits undergo a natural dynamics. For $n = 1, 2, \ldots$, let $f_n : \{0, 1\}^{m_n} \to \{0, 1\}$ be a Boolean function and $X^{(n)}(t) = (X_1(t), \ldots, X_{m_n}(t))_{t\in[0,\infty)}$ be a vector of i.i.d. stationary continuous time Markov chains on $\{0, 1\}$ that jump from 0 to 1 with rate $p_n \in [0, 1]$ and from 1 to 0 with rate $q_n = 1 - p_n$. Our object of study will be C_n which is the number of state changes of $f_n(X^{(n)}(t))$ as a function of t during $[0, 1]$. We say that the family $\{f_n\}_{n\geq1}$ is volatile if $C_n \to \infty$ in distribution as $n \to \infty$ and say that $\{f_n\}_{n\geq1}$ is tame if $\{C_n\}_{n\geq1}$ is tight. We study these concepts in and of themselves as well as investigate their relationship with the recent notions of noise sensitivity and noise stability. In addition, we study the question of lameness which means that $\mathbb{P}(C_n = 0) \to 1$ as $n \to \infty$. Finally, we investigate these properties for the majority function, iterated 3-majority, the AND/OR function on the binary tree and percolation on certain trees in various regimes.

MSC: 60K99

Keywords: Boolean function; Noise sensitivity; Noise stability

1. Introduction

We are given a sequence of Boolean functions $\{f_n\}_{n\geq1}$ with $f_n : \{0, 1\}^{m_n} \to \{0, 1\}$ for some sequence $\{m_n\}$ and also given a sequence $\{p_n\} \in [0, 1]$. Denoting $\{1, 2, \ldots, k\}$ by $[k]$, for each n

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and for each $i \in [m_n]$, let $\{X_i^{(n)}(t)\}_{t\in[0,\infty)}$ be the stationary continuous time Markov process on $\{0, 1\}$ that jumps from 0 to 1 with rate p_n and from 1 to 0 with rate $1 - p_n$ started in stationarity. (Equivalently $X_i^{(n)}(t)$ updates with rate 1 and at a given update, the value is chosen to be 1 with probability p_n and 0 with probability $1 - p_n$ independently of everything else.) Assume that the $\{X_i^{(n)}(t)\}_{t\in[0,\infty)}$ are independent as i varies and write $X^{(n)}(t)$ for $(X_1^{(n)}(t), \dots, X_{m_n}^{(n)}(t))$. Finally, the object of our focus will be $C_n([a, b])$ which is defined to be the number of times that $f_n(X^{(n)}(t))$ changes its state during the time interval $[a, b]$. We abbreviate $C_n([0, 1])$ by C_n throughout.

We say that $\{f_n\}$ is *degenerate* with respect to $\{p_n\}$ if

$$\lim_{n\to\infty} \mathbb{P}(f_n(X^{(n)}(0)) = 1)\mathbb{P}(f_n(X^{(n)}(0)) = 0) = 0$$

and *nondegenerate* with respect to $\{p_n\}$ if for some $\delta > 0$,

$$\mathbb{P}(f_n(X^{(n)}(0)) = 1) \in [\delta, 1 - \delta]$$

for all n. (Note that a sequence can of course be neither degenerate nor nondegenerate although it will always have a subsequence which is either one or the other.) The first concept we give captures the notion that it is unlikely that there is any change of state.

Definition 1.1. We say that $\{f_n\}_{n\ge1}$ is *lame with respect to* $\{p_n\}$ if

$$\lim_{n\to\infty} \mathbb{P}(C_n = 0) = 1.$$

The first relatively easy proposition says that a necessary condition for lameness is that the sequence is degenerate.

Proposition 1.2. *Let* $\{f_n\}_{n\ge1}$ *be a sequence of Boolean functions and* $\{p_n\}$ *be a sequence in* $[0, 1]$. *If* $\{f_n\}_{n\ge1}$ *is lame with respect to* $\{p_n\}$, *then it is degenerate with respect to* $\{p_n\}$.

The following two definitions will be central to the paper.

Definition 1.3. We say that $\{f_n\}_{n\ge1}$ is *volatile with respect to* $\{p_n\}$ if C_n approaches ∞ in distribution.

Definition 1.4. We say that $\{f_n\}_{n\ge1}$ is *tame with respect to* $\{p_n\}$ if $\{C_n\}_{n\ge1}$ is tight.

Note that lameness is a special case of tameness. While it is obvious that the notions of lameness, volatility and tameness may depend on the sequence $\{p_n\}$, it is natural to ask if these definitions depend on the length of the time interval chosen which we have taken to be 1. Lameness clearly does not. The fact that tameness does not depend on the length of the time interval is straightforward and follows from the fact that if a sequence of random vectors (X_n, Y_n) is such that $\{X_n\}_{n\ge1}$ and $\{Y_n\}_{n\ge1}$ are each tight, then $\{X_n + Y_n\}_{n\ge1}$ is also tight. The fact that volatility does not depend on the length of the time interval, while certainly believable and in fact true, does not follow from such general considerations. Rather, some explicit properties of the process are needed to establish this. It turns out that Markovianness and reversibility are sufficient. This follows from the following lemma, whose proof presented later is not so difficult.

Lemma 1.5. *Let* $\{f_n\}_{n\ge1}$ *be a sequence of Boolean functions and* $\{p_n\}$ *be a sequence in* $[0, 1]$. *The following four conditions are equivalent.*