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Stochastic Processes and their Applications 126 (2016) 3065–3076

stochastic processes and their applications

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Rate of convergence in first-passage percolation under low moments

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Received 22 September 2015; accepted 7 April 2016
Available online 27 April 2016

Abstract

We consider first-passage percolation on the d dimensional cubic lattice for $d \geq 2$; that is, we assign independently to each edge e a nonnegative random weight t_e with a common distribution and consider the induced random graph distance (the passage time), $T(x, y)$. It is known that for each $x \in \mathbb{Z}^d$, $\mu(x) = \lim_n T(0, nx)/n$ exists and that $0 \leq \mathbb{E}T(0, x) - \mu(x) \leq C\|x\|_1^{1/2} \log \|x\|_1$ under the condition $\mathbb{E}e^{\alpha t_e} < \infty$ for some $\alpha > 0$. By combining tools from concentration of measure with Alexander's methods, we show how such bounds can be extended to t_e's with distributions that have only low moments. For such edge-weights, we obtain an improved bound $C(\|x\|_1 \log \|x\|_1)^{1/2}$ and bounds on the rate of convergence to the limit shape.

Keywords: First-passage percolation; Concentration inequalities; Low moments; Rate of convergence

1. Introduction

1.1. The model

Let $d \geq 2$. Denote the set of nearest-neighbor edges of $\mathbb{Z}^d$ by $\mathcal{E}^d$, and let $(t_e)_{e \in \mathcal{E}^d}$ be a collection of non-negative random variables indexed by $\mathcal{E}^d$. For $x, y \in \mathbb{Z}^d$, define the passage time

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http://dx.doi.org/10.1016/j.spa.2016.04.001

$$\tau(x, y) = \inf_{\gamma: x \to y} \tau(\gamma),$$

where $\tau(\gamma) = \sum_{e \in \gamma} t_e$ and γ is any lattice path from x to y.

We will assume that $\mathbb{P}$, the distribution of (t_e), is a product measure satisfying the following conditions:

(A1) $\mathbb{E}Y^2 < \infty$, where Y is the minimum of d i.i.d. copies of t_e.
(A2) $\mathbb{P}(t_e = 0) < p_c$, where p_c is the threshold for d-dimensional bond percolation.

We now comment on assumptions (A1) and (A2). From Lemma 3.1 of [8], (A1) guarantees that $\mathbb{E}\tau(0, y)^{4-\eta} < \infty$ for all $\eta > 0$ and $y \in \mathbb{Z}^d$. Conversely, it is sufficient for (A1) that $\mathbb{E}t_e^{(2+\epsilon)/d}$ is finite for some $\epsilon > 0$. On the other hand, (A2) ensures that

$$\mathbb{P}(\exists \text{ a geodesic from } x \text{ to } y) = 1 \quad \text{for all } x, y \in \mathbb{Z}^d, \tag{1.1}$$

where a geodesic is a path γ from x to y that has $\tau(\gamma) = \tau(x, y)$. Under assumptions (A1) and (A2), equation 1.13 and Theorem 1.15 of [12] show that there exists a norm $\mu(\cdot)$ on $\mathbb{R}^d$, which is called the time constant, such that for $x \in \mathbb{Z}^d$, $\mathbb{P}$-almost surely,

$$\lim_{n\to\infty} \frac{1}{n}\tau(0, nx) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\tau(0, nx) = \inf_{n\geq 1} \frac{1}{n}\mathbb{E}\tau(0, nx) = \mu(x). \tag{1.2}$$

If (A1) is replaced by the condition that the minimum of $2d$ i.i.d. copies of t_e has finite dth moment, then the *shape theorem* holds; that is, for all $\epsilon > 0$, with probability one,

$$(1-\epsilon)B_0 \subset \frac{B(t)}{t} \subset (1+\epsilon)B_0 \quad \text{for all large } t, \tag{1.3}$$

where

$$B(t) := \left\{x + h; x \in \mathbb{Z}^d,\ \tau(0, x) \leq t,\ h \in \left[-\frac{1}{2}, \frac{1}{2}\right]^d\right\}$$

and $B_0 := \{x \in \mathbb{R}^d; \mu(x) \leq 1\}$, the *limit shape*.

1.2. Main results

Set $T = \tau(0, x)$. Our results below consist of (a) a bound on the deviation of $\mathbb{E}T$ from μ under (A1) and (A2) and (b) outer bounds on the rate of convergence to the limit shape under these same conditions and inner bounds under stronger conditions. These should be compared to the results of Alexander [2], who proved the first two with log in place of $\sqrt{\log}$ under exponential moments for t_e.

Proposition 1.1. *Assume* (A1) *and* (A2). *There exists* C_1 *such that for all* $x \in \mathbb{Z}^d$ *with* $\|x\|_1 > 1$,

$$\mu(x) \leq \mathbb{E}T \leq \mu(x) + C_1(\|x\|_1 \log \|x\|_1)^{1/2}.$$

Proof. This result follows by directly combining the Gaussian concentration inequality we derive below in Theorem 2.1 with Alexander's method of approximation of subadditive functions [2]. Our final bound is slightly better than the one given by Alexander in [2] because he used only an exponential concentration inequality. The interested reader can see the arXiv version of this paper [11] (version 1). □