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Properties of stochastic integro-differential equations with infinite delay: Regularity, ergodicity, weak sense Fokker–Planck equations

Hongwei Mei[a], George Yin[a,*], Fuke Wu[b]

[a] *Department of Mathematics, Wayne State University, Detroit, MI 48202, United States*
[b] *School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, PR China*

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Abstract

This work focuses on properties of stochastic integro-differential equations with infinite delay (or unbounded delay). Our main approach is to map the solution processes into another Polish space. Under suitable conditions, it is shown that the resulting processes are Markov. Furthermore, sufficient conditions for Feller properties, recurrence, ergodicity, and existence of invariant measures are obtained. Moreover, weak sense Fokker–Planck equations are derived for the underlying processes.

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1. Introduction

Time delays are omnipresent and virtually unavoidable in everyday life. Not only are they of great interest in the study of stochastic differential equations, but also they appear ubiquitously in

* Corresponding author.
E-mail addresses: hongwei.mei@wayne.edu (H. Mei), gyin@math.wayne.edu (G. Yin), wufuke@mail.hust.edu.cn (F. Wu).

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such applications as physics, biology, epidemics, transport, communication networks, and population models. Because of their importance, stochastic delay differential equations (SDDEs) and stochastic functional differential equations (SFDEs) have received considerable and sustained attention; see [2,3,5,7,11,16,19,17,20,24,23] and references therein.

Behaviors of stochastic systems with or without delays can be drastically different; see [14]. When we inject delay into a stochastic differential equation, the solution of the stochastic differential equation is no longer Markov. Losing Markovian properties makes things more difficult and complex. For a solution of a stochastic differential equation, we can connect it to a partial differential equation using Markovian property once we identify the operator of the diffusion process. However, when delay is present in a system, we no longer have an appropriate operator to work with. Thus, effort has been made to Markovianize the underlying systems when delay appears. To preserve the Markov property, Mohammed [20] examined a solution mapping instead of the solution process itself. The idea can be explained as follows. For a stochastic functional differential equation with finite delay (with delay range from $[-\tau, 0]$ for some finite $\tau > 0$)

$$dx(t) = f(x_t)dt + g(x_t)dw(t) \tag{1.1}$$

on $t \geq 0$, where $x_t = x_t(\theta) := \{x(t+\theta) \text{ with } -\tau \leq \theta \leq 0\}$, f and g satisfy appropriate conditions, and $w(t)$ is an m-dimensional standard Brownian motion. Under appropriate conditions, Mohammed [20] proved that although the solution process $x(t)$ is not Markov, the solution mapping x_t (more commonly referred to as segment process) possesses the Markov property. Moreover, he derived the exact form the weak infinitesimal generator associated with the process x_t.

It should be noted that many researchers have worked on asymptotic properties of systems with infinite delay (including both deterministic and stochastic) in the past decades; see for example, [1,4,9,15,18,24]. However, to the best of our knowledge, little is known for asymptotic properties such as ergodicity and invariant measures etc. of functional differential equations with infinite delay. This paper aims to fill in the gap.

To examine properties of stochastic delay or functional differential equations with infinite delay, denote by $\mathbb{BC} := BC((-\infty, 0], \mathbb{R}^d)$ the set of all $\mathbb{R}^d$-valued bounded and continuous function on $(-\infty, 0]$. For any $x \in \mathbb{BC}$, let π be an $\mathbb{R}$-valued function on $(-\infty, 0]$, and define

$$\Pi(x) = \Big(\int_{-\infty}^{0} \pi(s)x_1(s)ds, \ldots, \int_{-\infty}^{0} \pi(s)x_d(s)ds\Big).$$

Let $X(t)$ be a d-dimensional stochastic process satisfying the following stochastic integro-differential equation with non-random $x_0 \in \mathbb{BC}$. Consider the stochastic integro-differential equations with infinite delay

$$dX(t) = f(X(t), \Pi(X_t))dt + g(X(t), \Pi(X_t))dw(t) \tag{1.2}$$

with the initial data $X_0 = x_0$, where $X_t = \{X(t+s) : s \leq 0\}$ is the segment process of $X(t)$, $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$, and $w(t)$ is an m-dimensional standard Brownian Motion. Such equations have a wide range of applications. Take for instance, [12] considered a stochastic cellular neural networks model, which can be represented by the following stochastic integro-differential equations

$$du(t) = \Big[-Au(t) + Bf(u(t)) + Cf\Big(\int_0^\infty K(s)u(t-s)\Big)ds\Big]dt + \sigma(u(t))dw(t), \tag{1.3}$$