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Almost sure convergence of maxima for chaotic dynamical systems

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Abstract

Suppose $(f, \mathcal{X}, \nu)$ is a measure preserving dynamical system and $\phi : \mathcal{X} \to \mathbb{R}$ is an observable with some degree of regularity. We investigate the maximum process $M_n := \max(X_1, \dots, X_n)$, where $X_i = \phi \circ f^i$ is a time series of observations on the system. When $M_n \to \infty$ almost surely, we establish results on the almost sure growth rate, namely the existence (or otherwise) of a sequence $u_n \to \infty$ such that $M_n/u_n \to 1$ almost surely. For a wide class of non-uniformly hyperbolic dynamical systems we determine where such an almost sure limit exists and give examples where it does not.

Keywords: Limit theorems for dynamical systems; Extreme value theory

1. Introduction

Let $(f, \mathcal{X}, \nu)$ be a dynamical system, where $\mathcal{X} \subset \mathbb{R}$, $f : \mathcal{X} \to \mathcal{X}$ is a measurable transformation, and ν is an f-invariant probability measure supported on $\mathcal{X}$. Given an observable $\phi : \mathcal{X} \to \mathbb{R}$, i.e. a measurable function, we consider the stationary stochastic process $X_1, X_2, \dots$ defined as

$$X_i = \phi \circ f^{i-1}, \quad i \geq 1, \tag{1}$$

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and its associated maximum process M_n defined as

$$M_n = \max(X_1, \ldots, X_n). \tag{2}$$

Recent research has investigated the distributional behavior of M_n and in particular the existence of sequences $a_n, b_n \in \mathbb{R}$ such that

$$\nu\,(x \in \mathcal{X} : a_n(M_n - b_n) \leq u) \to G(u), \tag{3}$$

for some non-degenerate distribution function $G(u)$, $-\infty < u < \infty$. These results have shown that for sufficiently hyperbolic systems and for regular enough observables ϕ maximized at generic points $\tilde{x}$, the distribution limit is the same as that which would hold if (X_i) were independent identically distributed (i.i.d.) random variables with the same distribution function as ϕ [7,10,17,21]. There have been recent results on speed of convergence to the distributional limits, see [20,22,12]. As in the classical situation $G(u)$ can be one of the three extreme value distributions Type I (Gumbell), Type II (Frechét) and Type III (Weibull). If $\tilde{x}$ is periodic we expect different behavior (for details see [3,11,9,23]).

In [21, Lemma 1.1] the elementary observation is made:

Proposition 1.1. *Assume a function $x \mapsto g(x)$ has a minimum value of zero at a unique point $\tilde{x} \in \mathcal{X}$.*

The following are equivalent, where $\alpha > 0$:

1. *A Type I law for $x \mapsto -\log d(x, \tilde{x})$ with $a_n = 1$ and $b_n = \log n$;*
2. *A Type II law for $x \mapsto d(x, \tilde{x})^{-\alpha}$ with $a_n = n^{-\alpha}$ and $b_n = 0$;*
3. *A Type III law for $x \mapsto C - d(x, \tilde{x})^{\alpha}$ with $a_n = n^{\alpha}$ and $b_n = C$;*

(and similarly for other choices of b_n in the first case).

This paper will consider the almost sure behavior of M_n. A fundamental problem is to determine the existence (or otherwise) of a sequence $u_n \to \infty$ such that $M_n/u_n \to 1$ (almost surely) so as to determine almost sure rates of growth of M_n (perhaps with error bounds). Motivated by Proposition 1.1 and other applications we will consider the functions $-\log d(x, \tilde{x})$ and $d(x, \tilde{x})^{-\alpha}$. If ν is ergodic and ϕ is essentially bounded then almost surely, $M_n \to \operatorname{ess\,sup} \phi$, hence the limit of M_n in the case of a bounded observable (Type III) is clear.

For independent, identically distributed (i.i.d.) random variables the almost sure behavior of M_n has been widely studied, e.g. in the subject area of *extreme value theory* [8,13]. However within a dynamical systems framework, and also for general dependent random variables less is known about almost sure growth rates of M_n. There have been other results on the almost sure behavior of scaled Birkhoff sums of M_n in the i.i.d. case (see [5,29] and related references).

In this article we determine the existence (or otherwise) of sequences $u_n \to \infty$ such that $M_n/u_n \to 1$ for the functions $-\log d(x, \tilde{x})$ and $d(x, \tilde{x})^{-\alpha}$. In the case where $\phi(x) = -\log \operatorname{dist}(x, \tilde{x})$ for given $\tilde{x} \in \mathcal{X}$, we show for a broad class of chaotic systems that for ν-a.e. $\tilde{x} \in \mathcal{X}$

$$\lim_{n\to\infty} \frac{M_n(x)}{\log n} = \frac{1}{d_\nu}, \quad \text{a.s.,}$$

where d_ν is the local dimension of the measure ν. Towards proving almost sure convergence of M_n/u_n we will establish Borel–Cantelli results for non-uniformly hyperbolic systems and extend results of [19]. We will also consider observables of the form $\phi(x) = \operatorname{dist}(x, \tilde{x})^{-\alpha}$, $\alpha > 0$. In