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Modulation equation for SPDEs in unbounded domains with space–time white noise — Linear theory

Luigi Amedeo Bianchi*, Dirk Blömker

Institut für Mathematik, Universität Augsburg, D-86135 Augsburg, Germany

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Abstract

We study the approximation of SPDEs on the whole real line near a change of stability via modulation or amplitude equations, which acts as a replacement for the lack of random invariant manifolds on extended domains. Due to the unboundedness of the underlying domain a whole band of infinitely many eigenfunctions changes stability. Thus we expect not only a slow motion in time, but also a slow spatial modulation of the dominant modes, which is described by the modulation equation.

As a first step towards a full theory of modulation equations for nonlinear SPDEs on unbounded domains, we focus, in the results presented here, on the linear theory for one particular example, the Swift–Hohenberg equation. These linear results are one of the key technical tools to carry over the deterministic approximation results to the stochastic case with additive forcing. One technical problem for establishing error estimates rises from the spatially translation invariant nature of space–time white noise on unbounded domains, which implies that at any time we can expect the error to be always very large somewhere in space.

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* Corresponding author.

E-mail addresses: luigi.bianchi@math.uni-augsburg.de (L.A. Bianchi), dirk.bloemker@math.uni-augsburg.de (D. Blömker).

URLs: http://www.math.uni-augsburg.de/prof/ana/arbeitsgruppe/bianchi/ (L.A. Bianchi), http://www.math.uni-augsburg.de/prof/ana/arbeitsgruppe/bloemker/ (D. Blömker).

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1. Introduction

We study the approximation of stochastic partial differential equations (SPDEs) on unbounded domains near a change of stability of a trivial solution via modulation or amplitude equations. Due to the unboundedness of the underlying domain a whole infinite band (i.e., an interval) of eigenfunctions changes sign and therefore the trivial solution its stability. Thus neither the classical theory of invariant manifolds for PDEs nor the recently developed theory of random invariant manifolds [15,32,16,6,7] can be applied.

Modulation or amplitude equations are a replacement to overcome the lack of invariant manifolds, and they serve as a universal normal form depending only on the type of bifurcation. Being widely used in the physics literature, they are a tool to describe the evolution of the amplitude of the dominating pattern changing stability, where close to bifurcation we expect not only a slow motion of the amplitude in time, but also a slow modulation in space due to the band of eigenvalues changing sign.

For deterministic PDEs this theory is a well-established tool. See for example [8,26,36,29] for classical references, and the detailed comments later in this section. But hardly anything is known for SPDEs on unbounded domains.

As a starting point in this paper we consider the stochastic Swift–Hohenberg equation [10,22], which is a reduced model for the first convective instability in the Rayleigh–Bénard model and serves as one of the main examples in which pattern formation is studied. It is given as

$$\frac{\partial u}{\partial t} = -(1+\partial_x^2)^2 u + \varepsilon^2 \nu u - u^3 + \varepsilon^{3/2}\xi \tag{1}$$

on the whole real line with space–time white noise ξ. As we want to allow for periodic patterns, we do not assume any decay condition of solutions at infinity.

The theory of higher order parabolic stochastic partial differential equations (SPDEs) on unbounded domains with additive translation invariant noise like space–time white noise is not that well studied, while for the wave equation with multiplicative noise there are many recent publications (see for example [25,13,11,18]) and even more recent ones for parabolic equations with very rough noise [21,20].

In many cases parabolic equations with noise are studied subject to a spatial cut off or a decay condition at infinity. This is the case, for example, in [17], where the cut-off is both in the real space as well as in the Fourier space. Another example is [19]. In [5] and in a similar way in [27,28], the authors consider L^2-valued solutions, where an integral equation is consider instead of a PDE. The choice of trace class noise in these examples implies that we have an L^2-valued Wiener processes and thus a decay condition at infinity, which in both cases leads to more regular solutions.

If we were to consider decay at infinity, we conjecture we would recover similar results but with a point-forcing in the amplitude equation, due to the rescaling in space, needed to obtain the modulation equation.

The scaling of the equation involves small noise of order $\mathcal{O}(\varepsilon^{3/2})$ and small distance from bifurcation of order $\mathcal{O}(\varepsilon^2\nu)$. Due to the closeness to bifurcation, we expect small solutions and slow dynamics in time. Moreover, a whole band of Fourier modes around wave-number $k \pm 1$ changes sign close to $\mu = 0$, and thus we expect the dynamics to be given by a slow modulation of the complex amplitude A of the dominant pattern $e^{\pm ix}$:

$$u(t,x) \approx \varepsilon A(\varepsilon^2 t, \varepsilon x) \cdot e^{ix} + c.c.,$$