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The gap between Gromov-vague and Gromov–Hausdorff-vague topology

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Abstract

In Athreya et al. (2015) an invariance principle is stated for a class of strong Markov processes on tree-like metric measure spaces. It is shown that if the underlying spaces converge Gromov vaguely, then the processes converge in the sense of finite dimensional distributions. Further, if the underlying spaces converge Gromov–Hausdorff vaguely, then the processes converge weakly in path space. In this paper we systematically introduce and study the Gromov-vague and the Gromov–Hausdorff-vague topology on the space of equivalence classes of metric boundedly finite measure spaces. The latter topology is closely related to the Gromov–Hausdorff–Prohorov metric which is defined on different equivalence classes of metric measure spaces.

We explain the necessity of these two topologies via several examples, and close the gap between them. That is, we show that convergence in Gromov-vague topology implies convergence in Gromov–Hausdorff-vague topology if and only if the so-called lower mass-bound property is satisfied. Furthermore, we prove and disprove Polishness of several spaces of metric measure spaces in the topologies mentioned above.

As an application, we consider the Galton–Watson tree with critical offspring distribution of finite variance conditioned to not get extinct, and construct the so-called Kallenberg–Kesten tree as the weak limit in Gromov–Hausdorff-vague topology when the edge length is scaled down to go to zero.

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1. Introduction

The paper introduces the *Gromov-vague* and the *Gromov–Hausdorff-vague* topology. These are two notions of convergence of (equivalence classes of) metric boundedly finite measure spaces. These are "localized" versions of the Gromov-weak topology and a topology closely related to the Gromov–Hausdorff–Prohorov topology on (equivalence classes of) metric finite measure spaces.

Gromov-weak convergence and sampling. The Gromov-weak topology originates from the weak topology in the space of probability measures on a fixed metric space. It is an example of a topology which comes with a canonical family of measures and convergence determining test functions. That is, given a complete, separable metric space, (X, r), we denote by $\mathcal{M}_1(X)$ the space of all Borel probability measures on X and by $\bar{\mathcal{C}}(X) := \bar{\mathcal{C}}_{\mathbb{R}}(X)$ the space of bounded, continuous $\mathbb{R}$-valued functions. A sequence of probability measures (μ_n) converges *weakly* to μ in $\mathcal{M}_1(X)$ (abbreviated $\mu_n \Longrightarrow \mu$), as $n \to \infty$, if and only if $\int \mathrm{d}\mu_n\, f \to \int \mathrm{d}\mu\, f$ in $\mathbb{R}$, as $n \to \infty$, for all $f \in \bar{\mathcal{C}}(X)$.

We wish to consider sequence of measures that live on different spaces. In such a case an immediate analogue of bounded continuous functions is not available. To still be in a position to imitate the notion of weak convergence, we rely on the following useful fact: for a sequence (μ_n) in $\mathcal{M}_1(X)$ and $\mu \in \mathcal{M}_1(X)$,

$$\mu_n \underset{n\to\infty}{\Longrightarrow} \mu \quad \text{if and only if} \quad \mu_n^{\otimes\mathbb{N}} \underset{n\to\infty}{\Longrightarrow} \mu^{\otimes\mathbb{N}}. \tag{1.1}$$

Indeed, the *"if"* direction follows by the fact that projections to a single coordinate are continuous. The *"only if"* direction follows as the set of bounded continuous functions $\varphi\colon X^{\mathbb{N}} \to \mathbb{R}$ of the form $\varphi((x_n)_{n\in\mathbb{N}}) = \prod_{i=1}^{N} \varphi_i(x_i)$ for some $N \in \mathbb{N}, \varphi_i\colon X \to \mathbb{R}, i = 1, \ldots, N$, separates points in $X^{\mathbb{N}}$ and is multiplicatively closed (see, for example, [31, Theorem 2.7] for an argument how to use [29] to conclude from here that integration over such test functions is even convergence determining for measures on $\mathcal{M}_1(X)$).

Consider now the set of bounded continuous functions $\varphi\colon X^{\mathbb{N}} \to \mathbb{R}$ of the following form

$$\varphi = \tilde{\varphi} \circ R^{(X,r)}, \tag{1.2}$$

where $R^{(X,r)}$ denotes the map that sends a vector $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ to the matrix $(r(x_i, x_j))_{1\le i<j} \in \mathbb{R}_+^{\binom{\mathbb{N}}{2}}$ of mutual distances, and $\tilde{\varphi} \in \bar{\mathcal{C}}(\mathbb{R}_+^{\binom{\mathbb{N}}{2}})$ depends on finitely many coordinates only. A *(complete, separable) metric measure space* (X, r, μ) consists of a complete, separable metric space (X, r) and a Borel measure μ on X. Denote by $\mathbb{X}_1$ the space of measure preserving isometry classes of metric spaces equipped with a Borel probability measure. Then for each representative (X, r, μ) of an isometry class $\mathcal{X} \in \mathbb{X}_1$ the image measure $R_*^{(X,r)}\mu^{\otimes\mathbb{N}} = \mu^{\otimes\mathbb{N}} \circ (R^{(X,r)})^{-1} \in \mathcal{M}_1(\mathbb{R}_+^{\binom{\mathbb{N}}{2}})$ is the same and is referred to as the *distance matrix distribution* $\nu^{\mathcal{X}}$ of $\mathcal{X}$. It turns out that if the distance matrix distributions of two metric measure spaces coincide, then the metric measure spaces fall into the same isometry class. This is known as Gromov's reconstruction theorem (compare [23, Chapter $3\frac{1}{2}$]), and suggests to consider the *Gromov-weak topology*, which is the topology induced by the set of functions of the form

$$\Phi(X, r, \mu) = \int_{X^{\mathbb{N}}} \mathrm{d}\mu^{\otimes\mathbb{N}}\, \varphi = \int \mathrm{d}\nu^{\mathcal{X}}\, \tilde{\varphi}, \tag{1.3}$$