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Minimal supersolutions for BSDEs with singular terminal condition and application to optimal position targeting

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Abstract

We study the existence of a minimal supersolution for backward stochastic differential equations when the terminal data can take the value $+\infty$ with positive probability. We deal with equations on a general filtered probability space and with generators satisfying a general monotonicity assumption. With this minimal supersolution we then solve an optimal stochastic control problem related to portfolio liquidation problems. We generalize the existing results in three directions: firstly there is no assumption on the underlying filtration (except completeness and quasi-left continuity), secondly we relax the terminal liquidation constraint and finally the time horizon can be random.

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0. Introduction

This paper is devoted to the study of backward stochastic differential equations (BSDEs) with *singular* terminal condition. We adopt from [28,29] the notion of a weak (super) solution (Y, ψ, M) to a BSDE of the following form

$$dY_t = -f(t, Y_t, \psi_t)dt + \int_{\mathcal{Z}} \psi_t(z)\widetilde{\pi}(dz, dt) + dM_t, \tag{1}$$

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where $\widetilde{\pi}$ is a compensated Poisson random measure on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. The filtration $\mathbb{F}$ is supposed to be complete and right continuous. In particular, it can support a Brownian motion orthogonal to $\widetilde{\pi}$. The solution component M is required to be a local martingale orthogonal to $\widetilde{\pi}$. The function $f : \Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is called the *driver* (or *generator*) of the BSDE. The particularity here is that we allow the *terminal condition* ξ to be *singular*: for a stopping time τ, the random variable ξ is $\mathcal{F}_\tau$-measurable and takes the value $+\infty$ with positive probability.

In our first main result (Theorem 1) we establish existence of a *minimal* weak supersolution to (1). This supersolution is constructed via approximation from below. For each $L > 0$ we consider a truncated version of (1) with terminal condition $\xi \wedge L$. We impose that the driver f satisfies a monotonicity assumption in the y-variable and is Lipschitz continuous with respect to ψ. Then existence, uniqueness and comparison results for a solution (Y^L, ψ^L, M^L) to the truncated BSDE can be deduced from [23], where the theory of BSDEs with a monotone driver in a general filtration has been developed. We obtain the minimal supersolution (Y, ψ, M) with singular terminal condition by passing to the limit $L \to \infty$. The crucial task is to establish suitable a priori estimates for Y^L guaranteeing that when passing to the limit the solution Y does *not* explode before time τ. To this end, the generator f cannot be Lipschitz continuous w.r.t. y. Hence we impose that f is monotone and decreases at least polynomially with random coefficient in the y-variable. In the case where τ is deterministic this condition suffices to ensure boundedness of Y^L. When τ is random, we restrict attention to first exit of diffusions from a regular set.

BSDEs with singular terminal condition were already studied in [3,28] for deterministic terminal time (see also [12] for a treatise on BSPDEs), and in [29] for a random terminal time. Let us briefly outline in which directions our findings generalize some results from these papers.

- *General driver* f. Indeed, in the previously mentioned papers f is assumed to be a polynomial function of y (plus possibly a particular bounded from above function of ψ in [12]). Here f is supposed to be only bounded from above by a polynomial function w.r.t. y. The fact that we only assume here that f is Lipschitz continuous with respect to ψ but not necessarily bounded, requires to derive new a priori estimates for the family of solutions (Y^L). Moreover as in [3], the generator can be *singular* in the sense that the process $f_t^0 = f(t, 0, 0)$ can explode at time τ. We only impose an integrability condition on f^0 which is weaker than the condition in [3]. This weaker integrability condition and the occurrence of jumps imply that the convergence of the approximating sequence $(Y^L)_{L>0}$ has to be handled more carefully (see in particular the proof of Proposition 3 where technical details are postponed in the Appendix). BSDEs where the generator possesses a singularity in the time variable were studied in [19,18] to solve utility maximization problems with random horizon.
- *General filtration* $\mathbb{F}$. Moreover, compared to the papers [3,28,29], we do not restrict attention to a filtration generated by Brownian and Poisson noise. Here the filtration $\mathbb{F}$ satisfies only the standard assumptions (completeness and right-continuity). Hence the additional local martingale part M appears in the BSDE and has to be controlled when we let L go to $+\infty$. The quasi left-continuity condition on $\mathbb{F}$ will be imposed only to ensure the lower semi-continuity of Y at time τ: $\liminf_{t\to\tau} Y_t \geq \xi$.
- *Random terminal time* τ. To our best knowledge, [29] is the only paper that deals with a singular terminal condition at a random time τ. In this work, the generator f is equal to $f(y) = -y|y|^{q-1}$ for some $q > 1$ and the filtration is generated by a Brownian motion. When the terminal time is random, the derivation of the a priori estimate for the sequence Y^L