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Discretely sampled signals and the rough Hoff process

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Abstract

We introduce a canonical method for transforming a discrete sequential data set into an associated rough path made up of lead–lag increments. In particular, by sampling a d-dimensional continuous semimartingale $X : [0, 1] \to \mathbb{R}^d$ at a set of times $D = \{t_i\}$, we construct a piecewise linear, axis-directed process $X^D : [0, 1] \to \mathbb{R}^{2d}$ comprised of a past and a future component. We call such an object the Hoff process associated with the discrete data $\{X_t\}_{t_i \in D}$. The Hoff process can be lifted to its natural rough path enhancement and we consider the question of convergence as the sampling frequency increases. We prove that the Itô integral can be recovered from a sequence of random ODEs driven by the components of X^D. This is in contrast to the usual Stratonovich integral limit suggested by the classical Wong–Zakai Theorem (Wong and Zakai, 1965). Such random ODEs have a natural interpretation in the context of mathematical finance.

Keywords: Rough path theory; Lead–lag path; Hoff process; Wong–Zakai approximations; Itô–Stratonovich correction

1. Discrete streams of event data

A stream of event data is a time-ordered sequence of events where we have the time of the event as a real value and the data associated with the event typically taking some numerical value. In many applications, for example order books in finance or internet traffic, it is typical that these events come in a variety of quite different types at different frequencies, and that the realization of such streams can have a complex dependency structure between their coordinates. However, if

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we focus our interest in such event streams around their effects and the actions they trigger, then rough path theory provides an established and effective method (using the signature of the stream and the Itô map) to transform these event streams into much smaller data sets, while retaining adequate information to accurately approximate the primary effects of the data (see [19, §1–5]). The focus of rough path theory is to give quantitative mathematical meaning to the concept of the evolving response $y = (y_t)$ of a system f driven by a signal $x = (x_t)$, which we usually write in the form of a rough differential equation:

$$dy_t = f(y_t)\,dx_t, \quad y_0 = a. \tag{1.1}$$

Here, x can have a very complex local structure (that is, it can be a rough path). The theory shows how to summarize the data x while retaining a good approximation of y. A reader familiar with differential equations might want to think about the case where x is smooth but highly oscillatory on normal scales. The meaning of the differential equation (1.1) is not in doubt. Indeed, rough path theory tackles the question of how to define solutions by exploiting the smoothness of the system f. The theory summarizes x on normal scales so as to effectively capture its effects and predict y without knowing or caring further about x on microscopic scales.

This paper introduces a canonical method of transforming discrete numerical event data $\{X_{t_i}\}_{t_i \in D} \subset \mathbb{R}^d$ into a geometric 2-rough path $\mathbb{X}^D$ which we call a Hoff process. Importantly, our transformation takes into account the order of events and the effect of latency between different coordinates. Moreover, differential and integral equations driven by $\mathbb{X}^D$ have a classical as well as a rough path meaning, and we can consider the question of convergence as our sampling rate becomes finer by exploiting the Itô map from rough path theory.

With this in mind, this paper examines the interplay between this way of looking at discrete data through the Hoff process in the special case of when the discrete event data comes from a multidimensional semimartingale signal. In effect, we prove the equivalent of the Wong–Zakai Theorem by demonstrating that, under certain regularity conditions, a sequence of Hoff processes is Cauchy in the standard p-variation rough path metric as the sampling frequency becomes finer. This is the content of the first main result of the paper (Theorem 4.1). Before introducing the second result, let us formally define a Hoff process in the next section.

2. Defining the Hoff process

This section defines the Hoff process as a method of transforming discrete numerical event data in $\mathbb{R}^d$ into a geometric 2-rough path. Given a sample $\{X_{t_i}\}_{t_i \in D} \subset \mathbb{R}^d$ at times $D = \{t_i\}_{i=0}^n \subset [0, 1]$, the standard approach to transforming this into a continuous time trajectory is to define the càdlàg, piecewise constant path $Z : [0, 1] \to \mathbb{R}^d$ by $Z_t = X_{t_i}$ for $t \in [t_i, t_{i+1})$. Although this is a natural construction, the order of events across different coordinates is in some sense lost. Following the Ph.D. thesis [15] of B. Hoff, we go one step further and consider the continuous process made up of lead and lag versions of this standard interpolation. We call the resultant path $X^D : [0, 1] \to \mathbb{R}^{2d}$ the Hoff process:

Definition 2.1. Let $D = \{t_i\}_{i=0}^n \subset [0, 1]$ be a sequence of times with $(t_0, t_n) = (0, 1)$. Define the path $\widehat{X}_t : [0, 2n] \to \mathbb{R}^{2d}$ given by

$$\widehat{X}_t^D = \left(\widehat{X}_t^{D,b}, \widehat{X}_t^{D,f}\right)$$