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Muckenhoupt's (A_p) condition and the existence of the optimal martingale measure

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Abstract

In the problem of optimal investment with a utility function defined on $(0, \infty)$, we formulate sufficient conditions for the dual optimizer to be a uniformly integrable martingale. Our key requirement consists of the existence of a martingale measure whose density process satisfies the probabilistic Muckenhoupt (A_p) condition for the power $p = 1/(1 - a)$, where $a \in (0, 1)$ is a lower bound on the relative risk-aversion of the utility function. We construct a counterexample showing that this (A_p) condition is sharp.

MSC: 60G44; 91G10

Keywords: Utility maximization; Optimal martingale measure; BMO martingales; (A_p) condition

1. Introduction

An unpleasant qualitative feature of the general theory of optimal investment with a utility function defined on $(0, \infty)$ is that the dual optimizer $\widehat{Y}$ may not be a uniformly integrable

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martingale. In the presence of jumps, it may even fail to be a local martingale. The corresponding counterexamples can be found in [16]. In this paper, we seek to provide conditions under which the uniform martingale property for $\widehat{Y}$ holds and thus, $\widehat{Y}/\widehat{Y}_0$ defines the density process of the *optimal* martingale measure $\widehat{\mathbb{Q}}$.

The question of whether $\widehat{Y}$ is a uniformly integrable martingale is of longstanding interest in mathematical finance and can be traced back to [9,13]. This problem naturally arises in situations involving utility-based arguments. For instance, it is relevant for pricing in incomplete markets, where according to [11] the existence of $\widehat{\mathbb{Q}}$ is equivalent to the fact that for every bounded contingent claim ψ its marginal utility-based price p is unique. In this case,

$$p = \mathbb{E}^{\widehat{\mathbb{Q}}}[\psi] = \mathbb{E}\left[\frac{\widehat{Y}_T}{\widehat{Y}_0}\psi\right],$$

and thus $\widehat{\mathbb{Q}}$ plays the role of the pricing measure from the classical Black and Scholes theory of complete financial markets, see [20,2]. Notice that the nonexistence of $\widehat{\mathbb{Q}}$ is equivalent to $\mathbb{E}\left[\frac{\widehat{Y}_T}{\widehat{Y}_0}\right] < 1$. Then for $\psi = 1$ the expression $\mathbb{E}\left[\frac{\widehat{Y}_T}{\widehat{Y}_0}\psi\right]$ fails to be even an arbitrage-free price!

Of course, if the dual minimizer $\widehat{Y}$ can be computed *explicitly* as in [15], then its uniform integrability property may be verified using either the sufficient conditions of Novikov and Kazamaki or the necessary and sufficient criteria based on Hellinger processes. We refer the reader to [14, Section 1.4] for the former and to [12, Section IV.2] for the latter. However, for a generic incomplete model there is little hope of obtaining an explicit representation for $\widehat{Y}$, and a different approach should be used.

Our key requirement consists of the existence of a dual supermartingale Z, which satisfies the probabilistic Muckenhoupt (A_p) condition for the power $p > 1$ such that

$$p = \frac{1}{1-a}. \tag{1.1}$$

Here $a \in (0, 1)$ is a lower bound on the relative risk-aversion of the utility function. As we prove in Theorem 5.1, this condition, along with the existence of an upper bound for the relative risk-aversion, yields $(A_{p'})$ for $\widehat{Y}$ for some $p' > 1$. This property in turn implies that the dual minimizer $\widehat{Y}$ is of class (**D**), that is, the family of its values evaluated at all stopping times is uniformly integrable. In Proposition 6.1, we construct a counterexample showing that the bound (1.1) is the best possible for $\widehat{Y}$ to be of class (**D**) even in the case of power utilities and continuous stock prices.

In the case of the power utility function

$$U(x) = \frac{x^{1-a}}{1-a}, \quad x > 0,$$

with the relative risk-aversion $a \in (0, 1)$ the dual optimizer $\widehat{Y}$ satisfies (A_p) with p given by (1.1) if and only if the (A_p) condition holds for some dual supermartingale Z. Moreover, $\widehat{Y}$ has the smallest (A_p)-constant among all such Z. This fact has been already established in [19]. For reader's convenience we shall restate it as Proposition 3.3.

A similar idea of passing regularity from some dual element to the optimal one has been employed in [4,8,3] for respectively, quadratic, power and exponential utility functions defined on the whole real line. These papers use appropriate versions of the Reverse Hölder (R_q) inequality. We recall that (A_p) and (R_q) conditions are dual in the sense that if Z is the density process of the