Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 126 (2016) 2665–2718

stochastic processes and their applications

www.elsevier.com/locate/spa

Viscosity solutions of path-dependent integro-differential equations

Christian Keller

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, United States

Received 28 March 2015; received in revised form 8 February 2016; accepted 25 February 2016
Available online 7 March 2016

Abstract

We extend the notion of viscosity solutions for path-dependent PDEs introduced by Ekren et al. (2014) to path-dependent integro-differential equations and establish well-posedness, i.e., existence, uniqueness, and stability, for a class of semilinear path-dependent integro-differential equations with uniformly continuous data. Closely related are non-Markovian backward SDEs with jumps, which provide a probabilistic representation for solutions of our equations. The results are potentially useful for applications using non-Markovian jump–diffusion models.

MSC: 45K05; 35D40; 60H10; 60H30

Keywords: Path-dependent integro-differential equations; Viscosity solutions; Backward SDEs with jumps; Skorokhod topologies; Martingale problems

1. Introduction

The goal of this paper is to extend the theory of viscosity solutions (in the sense of [18,19]) for path-dependent partial differential equations (PPDEs) to path-dependent integro-differential equations. In particular, we investigate semilinear path-dependent integro-differential equations

E-mail address: ckell@umich.edu.

http://dx.doi.org/10.1016/j.spa.2016.02.014
0304-4149/

of the form

$$\mathcal{L}u(t,\omega) - f_t(\omega, u(t,\omega), \partial_\omega u(t,\omega), \mathcal{I}u(t,\omega)) = 0,$$
$$(t,\omega) \in [0,T) \times \mathbb{D}([0,T],\mathbb{R}^d), \tag{1.1}$$

where $\mathbb{D}([0,T],\mathbb{R}^d)$ is the space of right-continuous functions with left limits from $[0,T]$ to $\mathbb{R}^d$, $\mathcal{L}$ is a linear integro-differential operator of the form

$$\mathcal{L}u(t,\omega) = -\partial_t u(t,\omega) - \sum_{i=1}^d b_t^i(\omega)\partial_{\omega^i} u(t,\omega) - \frac{1}{2}\sum_{i,j=1}^d c_t^{ij}(\omega)\partial^2_{\omega^i\omega^j} u(t,\omega)$$
$$- \int_{\mathbb{R}^d} \left[u(t,\omega + z\cdot \mathbf{1}_{[t,T]}) - u(t,\omega) - \sum_{i=1}^d z^i\, \partial_{\omega^i} u(t,\omega)\right] K_t(\omega, dz),$$

and $\mathcal{I}$ is an integral operator of the form

$$\mathcal{I}u(t,\omega) = \int_{\mathbb{R}^d} \left[u(t,\omega + z\cdot \mathbf{1}_{[t,T]}) - u(t,\omega)\right] \eta_t(\omega,z)\, K_t(\omega,dz).$$

Well-posedness for semilinear PPDEs has been first established by Ekren, Keller, Touzi, and Zhang [18], where also the here used notion of viscosity solutions has been introduced. Subsequent work by Ekren, Touzi and Zhang deals with fully nonlinear PPDEs [19,20], by Pham and Zhang with path-dependent Isaacs equations [39], by Ekren with obstacle PPDEs [17], and by Ren with fully nonlinear elliptic PPDEs [40].

Initial motivation for this line of research came from Peng [38], who considered non-Markovian backward stochastic differential equations (BSDEs) as PPDEs analogously to the relationship between Markovian BSDEs and (standard) PDEs, from Dupire [16], who introduced new derivatives on $\mathbb{D}([0,T],\mathbb{R}^d)$ so that for smooth functionals on $[0,T) \times \mathbb{D}([0,T],\mathbb{R}^d)$ a functional counterpart to Itôs formula holds, and from Cont and Fournié (see [8–10]), who extended Dupire's seminal work. However, fully nonlinear PPDEs of first order have been studied earlier by Lukoyanov (see, for example, [30–32]). He used derivatives introduced by Kim [29] and adapted first the notion of so-called minimax solution and then of viscosity solutions from PDEs to PPDEs. Minimax solutions for PDEs have been introduced by Subbotin (see, e.g., [43, 44]) motivated by the study of differential games. In the case of PDEs of first order, minimax and viscosity solutions are equivalent (see [45]). Another approach for generalized solution for first-order PPDEs can be found in work by Aubin and Haddad [1], where so-called Clio derivatives for path-dependent functionals are introduced in order to study certain path-dependent Hamilton–Jacobi–Bellman equations that occur in portfolio theory.

Possible applications of path-dependent integro-differential equations are non-Markovian problems in control, differential games, and financial mathematics that involve jump processes.

Some comments about differences between PDEs and PPDEs seem to be in order. Contrary to standard PDEs, even linear PPDEs have rarely classical solutions in most relevant situations. Hence, one needs to consider a weaker form of solutions. In the case of PDEs, the notion of viscosity solutions introduced by Crandall and Lions [11] turned out to be extremely successful. The main difficulty in the path-dependent case compared to the standard PDE case is the lack of local compactness of the state space, e.g., $[0,T] \times \mathbb{D}$ vs. $[0,T] \times \mathbb{R}^d$. Local compactness is essential for proofs of uniqueness of viscosity solutions to PDEs, i.e., PDE standard methods can, in general, not easily adapted to the path-dependent case. The main contribution of [18] was to replace the pointwise supremum/infimum occurring in the definition of viscosity solutions to