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Convergence, unanimity and disagreement in majority dynamics on unimodular graphs and random graphs

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Abstract

In majority dynamics, agents located at the vertices of an undirected simple graph update their binary opinions synchronously by adopting those of the majority of their neighbors.

On infinite unimodular transitive graphs we show that the opinion of each agent almost surely either converges, or else eventually oscillates with period two; this is known to hold for finite graphs, but not for all infinite graphs.

On Erdős–Rényi random graphs with degrees $\Omega(\sqrt{n})$, we show that agents eventually all agree, with constant probability. Conversely, on random 4-regular finite graphs, we show that with high probability different agents converge to different opinions.

Keywords: Majority dynamics; Random graphs

1. Introduction

Let $G = (V, E)$ be a finite or countably infinite, locally finite, undirected simple graph. Consider time periods $t \in \{0, 1, 2, \ldots\}$ and, for each time t and $i \in V$, let $X_t(i) \in \{-1, +1\}$ be the opinion of vertex i at time t.

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We define *majority dynamics* by

$$X_{t+1}(i) = \operatorname{sgn} \sum_{j \in \partial(i)} X_t(j), \tag{1.1}$$

where $\partial(i)$ is the set of neighbors of i in G. To resolve (or avoid) ties, we either add or remove i from $\partial(i)$ so that $|\partial(i)|$ is odd. This ensures that the sum in the r.h.s. of (1.1) is never zero. Equivalently, we let ties be broken by reverting to the agent's existing opinion.

A well known result is the *period two property* of finite graphs, due to Goles and Olivos [8].

Theorem 1.1 (*Goles and Olivos*). *For every finite graph $G = (V, E)$, initial opinions $\{X_0(i)\}_{i \in V}$ and vertex i it holds that $X_{t+2}(i) = X_t(i)$ for all sufficiently large t.*

That is, every agent's opinion eventually converges, or else enters a cycle of length two.

This theorem also holds for some infinite graphs [11,7]; in particular for those of bounded degree and subexponential growth, or slow enough exponential growth. In [16] it is furthermore shown that on graphs of maximum degree d the number of times t for which $X_{t+2}(i) \neq X_t(i)$ is at most

$$\frac{d+1}{d-1} \cdot d \cdot \sum_{r=0}^{\infty} \left(\frac{d+1}{d-1}\right)^{-r} n_r(G, i),$$

where $n_r(G, i)$ is the number of vertices at graph distance r from i in G.

However, on some infinite graphs there exist initial configurations of the opinions such that no agent's opinion converges to any period; this is easy to construct on regular trees. A natural question is whether such configurations are "rare", in the sense that they appear with probability zero for some natural probability distribution on the initial configurations. In [9] it was shown that on a regular trees, when initial opinions are chosen i.i.d. with sufficient bias towards $+1$, then all opinions converge to $+1$ with probability one. It was shown also that this is not the case in some odd degree regular trees, when the bias is sufficiently small. However, the question of whether opinions converge at all when the bias is small was not addressed.

We show that indeed opinions almost surely converge (or enter a cycle with period two) on regular trees, whenever the initial configuration is chosen i.i.d. In fact, we prove a much more general result.

A *graph isomorphism* between graphs $G = (V, E)$ and $G' = (V', E')$ is a bijection $h : V \to V'$ such that $(i, j) \in E$ iff $(h(i), h(j)) \in E'$. Intuitively, two graphs are isomorphic if they are equal, up to a renaming of the vertices.

The *automorphism group* $\operatorname{Aut}(G)$ is the set of isomorphisms from G to G, equipped with the operation of composition. G is said to be *transitive* if $\operatorname{Aut}(G)$ acts transitively on V. That is, if there is a single orbit V/G, or, equivalently, if for every $i, j \in V$ there exists an $h \in \operatorname{Aut}(G)$ such that $h(i) = j$. G is said to be *unimodular* if $\operatorname{Aut}(G)$ is unimodular (see, e.g., Aldous and Lyons [1]).[1] G is unimodular if and only i the following "mass transport principle" holds: informally, in every flow on the graph that is invariant to $\operatorname{Aut}(G)$, the sum of what flows into a node is equal to the sum of what flows out. Formally, for every $F\colon V \times V \to \mathbb{R}^+$ that is invariant with respect to the diagonal action of $\operatorname{Aut}(G)$ it holds that

$$\sum_{j \in \partial(i)} f(i, j) = \sum_{j \in \partial(i)} f(j, i),$$

where $i \in V$ is arbitrary.

[1] See [17] for an example (Trofimov's "grandfather graph") of a transitive graph that is not unimodular.