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# An explicit martingale version of the one-dimensional Brenier's Theorem with full marginals constraint[☆]

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**Abstract**

We provide an extension of the martingale version of the Fréchet–Hoeffding coupling to the infinitely-many marginals constraints setting. In the two-marginal context, this extension was obtained by Beiglböck and Juillet (2016), and further developed by Henry-Labordère and Touzi (in press), see also Beiglböck and Henry-Labordère (Preprint).

Our main result applies to a special class of reward functions and requires some restrictions on the marginal distributions. We show that the optimal martingale transference plan is induced by a pure downward jump local Lévy model. In particular, this provides a new martingale peacock process (PCOC "Processus Croissant pour l'Ordre Convexe," see Hirsch et al. (2011), and a new remarkable example of discontinuous fake Brownian motions. Further, as in Henry-Labordère and Touzi (in press), we also provide a duality result together with the corresponding dual optimizer in explicit form.

As an application to financial mathematics, our results give the model-independent optimal lower and upper bounds for variance swaps.


*Keywords:* Martingale optimal transport, Brenier's Theorem; PCOC; Fake Brownian motion

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## 1. Introduction

The classical optimal transport (OT) problem was initially formulated by Monge in his treatise "Théorie des déblais et des remblais" as follows. Let $\mu_0, \mu_1$ be two probability measures on $\mathbb{R}^d$, $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a cost function, then the optimal transport problem consists in minimizing the cost $\int_{\mathbb{R}^d} c(x, T(x))\mu_0(dx)$ among all transference plans, i.e. all measurable functions $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $\mu_1 = \mu_0 \circ T^{-1}$. The relaxed formulation of the problem, as introduced by Kantorovich, consists in minimizing the value $\mathbb{E}^{\mathbb{P}}[c(X_0, X_1)]$ among all probability measures $\mathbb{P}$ such that $\mathbb{P} \circ X_0^{-1} = \mu_0$ and $\mathbb{P} \circ X_1^{-1} = \mu_1$. Under the so-called Spence–Mirrlees or Twist condition, the optimal Monge transference plan is characterized by the Brenier Theorem, and explicitly given by the Fréchet–Hoeffding in the one-dimensional setting. We refer to Rachev and Rüschendorf [67] and Villani [71] for a detailed presentation.

The theory has been extended to the multiple marginals case by Gangbo and Święch [31], Carlier [15], Olkin and Rachev [61], Knott and Smith [56], Rüschendorf and Uckelmann [68], Heinich [36], and Pass [63,64,66], etc. We also refer to the full-marginals case addressed by Pass [65].

Recently, a martingale transportation (MT) problem was introduced in Beiglböck, Henry-Labordère and Penkner [5] and in Galichon, Henry-Labordère and Touzi [30]. Given two probability measures $\mu_0$ and $\mu_1$, the problem consists in minimizing some expected cost among all probability measures $\mathbb{P}$ with fixed marginals $\mathbb{P} \circ X_0^{-1} = \mu_0$, $\mathbb{P} \circ X_1^{-1} = \mu_1$, and such that the canonical process $X$ is a $\mathbb{P}$-martingale.

This new optimal transport problem is motivated by the problem of robust subhedging exotic options in a frictionless financial market allowing for trading the underlying asset and the corresponding vanilla options for the maturities 0 and 1. As observed by Breeden and Litzenberger [12], the market values of vanilla options for all strikes allow to recover the marginal distributions of the underlying asset price. This suggests a dual formulation of the robust superhedging problem defined as the minimization of the $\mathbb{P}$-expected payoff of the exotic option over all martingale measures $\mathbb{P}$ satisfying the marginal distribution constraint.

Based on the fact that any martingale can be represented as a time-changed Brownian motion, this problem was initially studied in the seminal paper of Hobson [42] by means of the Skorokhod Embedding Problem (SEP) approach, which consists in finding a stopping time $\tau$ of Brownian motion $B$ such that $B_\tau$ has some given distribution. This methodology generated developments in many directions, namely for different derivative contracts and/or multiple-marginals constraints, see e.g. Brown, Hobson and Rogers [13], Madan and Yor [58], Cox, Hobson and Obłój [17], Cox and Obłój [18,19], Davis, Obłój and Raval [22], Cox and Wang [21], Gassiat, Oberhauser and dos Reis [32], Cox, Obłój and Touzi [20], Hobson and Neuberger [49], and Hobson and Klimmek [46,48,47,45]. We also refer to the survey papers by Obłój [59] and Hobson [43] for more details.

Recently, a rich literature has emerged around the martingale optimal transport approach to robust hedging. For models in discrete-time, we refer to Acciaio, Beiglböck, Penkner and Schachermayer [1], Beiglböck and Nutz [8], Beiglböck, Henry-Labordère and Touzi [6], Bouchard and Nutz [11], Campi, Laachir and Martini [14], Fahim and Huang [27], De Marco and Henry-Labordère [23]. For models in continuous-time, we refer to Biagini, Bouchard, Kardaras and Nutz [9], Dolinsky and Soner [24–26], Juillet [52], Henry-Labordère, Obłój, Spoida and Touzi [37], Källblad, Tan and Touzi [53], Stebegg [69], Bonnans and Tan [10], and Tan and Touzi [70]. We finally mention the work by Beiglböck, Cox and Huesmann [3] which derives