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Stochastic Processes and their Applications 126 (2016) 2860–2875
stochastic processes and their applications
www.elsevier.com/locate/spa

On weak convergence of stochastic heat equation with colored noise

Pavel Bezdek

Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090, United States

Received 4 August 2015; received in revised form 15 January 2016; accepted 10 March 2016
Available online 24 March 2016

Abstract

In this work we are going to show weak convergence of probability measures. The measure corresponding to the solution of the following one dimensional nonlinear stochastic heat equation $\frac{\partial}{\partial t}u_t(x) = \frac{\kappa}{2}\frac{\partial^2}{\partial x^2}u_t(x) + \sigma(u_t(x))\eta_\alpha$ with colored noise η_α will converge to the measure corresponding to the solution of the same equation but with white noise η, as $\alpha \uparrow 1$. Function σ is taken to be Lipschitz and the Gaussian noise η_α is assumed to be colored in space and its covariance is given by $\mathrm{E}[\eta_\alpha(t,x)\eta_\alpha(s,y)] = \delta(t-s)f_\alpha(x-y)$ where f_α is the Riesz kernel $f_\alpha(x) \propto 1/|x|^\alpha$. We will work with the classical notion of weak convergence of measures, that is convergence of probability measures on a space of continuous function with compact domain and sup–norm topology. We will also state a result about continuity of measures in α, for $\alpha \in (0, 1)$.
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Keywords: The stochastic heat equation; Colored noise; Riesz kernel

1. Introduction

Throughout this work we will consider the following one-dimensional heat equation

$$\frac{\partial}{\partial t}u_{\alpha,t}(x) = \frac{\kappa}{2}\frac{\partial^2}{\partial x^2}u_{\alpha,t}(x) + \sigma(u_{\alpha,t}(x))\eta_\alpha, \quad x \in \mathbb{R},\ t \geq 0, \tag{1}$$
$$u_{\alpha,0} = w(x),$$

E-mail address: bezdek@math.utah.edu.

http://dx.doi.org/10.1016/j.spa.2016.03.006
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with $\kappa > 0$ and Gaussian space time colored noise η_α [6]. The noise η_α is assumed to have a particular covariance structure

$$\mathrm{E}\left[\eta_\alpha(t,x)\eta_\alpha(s,y)\right] = \delta(t-s) f_\alpha(x-y), \tag{2}$$

where [6, Ex. 1]

$$f_\alpha(x) = c_{1-\alpha} g_\alpha(x) = \hat{g}_{1-\alpha}(x), \qquad g_\alpha(x) = \frac{1}{|x|^\alpha} \quad \text{for } \alpha \in (0,1), \tag{3}$$

and the constant c_α is [9, (12) on pg. 173]

$$c_\alpha = 2\frac{\sin\left(\frac{\alpha\pi}{2}\right) \Gamma(1-\alpha)}{(2\pi)^{1-\alpha}}. \tag{4}$$

The function $\hat{g}_{1-\alpha}$ denotes the Fourier of function $g_{1-\alpha}$. For $F \in L^1(\mathbb{R})$, we will take $\hat{F}(\xi) = \int_{\mathbb{R}} e^{-2\pi\xi x} F(x)dx$. The initial condition, $w(x)$ is taken to be bounded and ϱ-Hölder continuous. We will also assume σ to be Lipschitz continuous, there exists $K \geq 0$ such that $|\sigma(x) - \sigma(y)| \leq K|x-y|$ and $|\sigma(x)| \leq K(1+|x|)$. Stochastic PDEs such as (1) have been studied in [6,14,2,13,5] and others.

The function f_α can be thought of as an *'approximation'* to the delta function in the following special sense, we know that one-dimensional Fourier transform of $g_{1-\alpha}$, denoted by $\hat{g}_{1-\alpha}$, is equal to f_α. We also know that the Fourier transform of a constant is δ distribution. Observe that $g_{1-\alpha}$ converges pointwise to 1 as $\alpha \uparrow 1$. We will study the solution of (1) as a function of α. This arises noticeably in [1, Sec. 7] where the authors have shown that $L^2(P)$ norm of $u_{\alpha,t}(x)$ converges to $L^2(P)$ norm of the solution to (5) as $\alpha \uparrow 1$ for every $t > 0$, $x \in \mathbb{R}$ and $\sigma(x) = x$.

The main question that has motivated this work, is whether the solution of (1) converges *in the appropriate sense* to the solution of the same equation, but with white noise η instead of colored noise η_α as $\alpha \uparrow 1$. By that we mean, the solution to

$$\begin{aligned} \frac{\partial}{\partial t} u_t(x) &= \frac{\kappa}{2}\frac{\partial^2}{\partial x^2} u_t(x) + \sigma(u_t(x))\eta, \quad x \in \mathbb{R},\ t \geq 0, \\ u_0(x) &= w(x), \end{aligned} \tag{5}$$

where η denotes white noise. We will state the main theorem in terms of measures corresponding to solutions. Let $\mathcal{C} = \mathcal{C}([0,T] \times [-N,N])$ be the space of continuous functions on $[0,T] \times [-N,N] \subset \mathbb{R}^+ \times \mathbb{R}$ with supremum norm. Denote by P_α, the measure corresponding to u_α restricted to $D = [0,T] \times [-N,N]$,

$$\mathrm{P}_\alpha(A) := \begin{cases} \mathrm{P}\left\{u_\alpha \in A^\circ\right\} & \text{for } \alpha \in (0,1), \\ \mathrm{P}\left\{u \in A^\circ\right\} & \text{for } \alpha = 1, \end{cases}$$

for any Borel set A of space $\mathcal{C}$. By A°, we denote the embedding of the set A in a larger space $\mathcal{C}(\mathbb{R}_+ \times \mathbb{R})$, that is

$$A^\circ = \{f \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}) : f \text{ restricted to } [0,T] \times [-N,N] \text{ is in } A\}.$$

Here is the main theorem:

Theorem 1. *Measure* P_α *is continuous in* α*, for* $\alpha \in (0,1]$. *We precisely mean that* P_α *converges weakly to* P_1 *as* $\alpha \uparrow 1$ *and* P_α *converges weakly to* P_{α_0} *as* $\alpha \to \alpha_0$ *for any* $\alpha_0 \in (0,1)$.

The notion of weak convergence in Theorem 1 is the classical one [4]. Theorem 1 gives us a new way of thinking about the stochastic heat equation with white noise. Instead of studying

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