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On the functional CLT for stationary Markov chains started at a point

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Dedicated to the memory of Mikhail Gordin

Abstract

We present a general functional central limit theorem started at a point also known under the name of quenched. As a consequence, we point out several new classes of stationary processes, defined via projection conditions, which satisfy this type of asymptotic result. One of the theorems shows that if a Markov chain is stationary ergodic and reversible, this result holds for bounded additive functionals of the chain which have a martingale coboundary in $\mathbb{L}_1$ representation. Our results are also well adapted for strongly mixing sequences providing for this case an alternative, shorter approach to some recent results in the literature.

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Keywords: Functional central limit theorem; Quenched convergence; Functions of Markov chains; Martingale approximation; Reversible Markov chains

1. Introduction and results

In this paper we address the question of the validity of functional limit theorem for processes started at a point for almost all starting points. These types of results are also known under the

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name of quenched limit theorems or almost sure conditional invariance principles. The quenched functional CLT is more general than the usual one and it is very important for analyzing random processes in random environment, Markov chain Monte Carlo procedures and the discrete Fourier transform (see [30,31,2]). On the other hand there are numerous examples of processes satisfying the functional CLT but failing to satisfy the quenched CLT. Some examples were constructed by Volný and Woodroofe [35] and for the discrete Fourier transforms by Barrera [1]. This is the reason why it is desirable to point out classes of processes satisfying a quenched CLT. Special attention will be devoted to reversible Markov chains and several open problems will be pointed out. Reversible Markov chains have applications to statistical mechanics and to Metropolis Hastings algorithms used in Monte Carlo simulations. The methods of proof we used are based on martingale techniques combined with results from ergodic theory.

The field of limit theorems for stationary stochastic processes is closely related to Markov operators and dynamical systems. All the results for stationary sequences can be translated in the language of Markov operators and vice-versa. In this paper we shall mainly use the Markov operator language and also indicate the connection with stationary processes.

We assume that $(\xi_n)_{n\in\mathbb{Z}}$ is a stationary Markov chain defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a measurable state space $(S, \mathcal{A})$, with marginal distribution $\pi(A) = \mathbb{P}(\xi_0 \in A)$ and regular conditional distribution for ξ_1 given ξ_0, denoted by $Q(x, A) = \mathbb{P}(\xi_1 \in A|\xi_0 = x)$. Let Q also denote the Markov operator acting via $(Qf)(x) = \int_S f(s)Q(x, ds)$. Next, for $p \geq 1$, let $\mathbb{L}_p^0(\pi)$ be the set of measurable functions on S such that $\int |f|^p d\pi < \infty$ and $\int f d\pi = 0$. For some function $f \in \mathbb{L}_2^0(\pi)$, let

$$X_i = f(\xi_i), \qquad S_n = S_n(f) = \sum_{i=1}^{n} X_i. \tag{1}$$

Denote by $\mathcal{F}_k$ the σ-field generated by ξ_i with $i \leq k$. For any integrable random variable X we denote by $\mathbb{E}_k(X) = \mathbb{E}(X|\mathcal{F}_k)$ the conditional expectation of X given $\mathcal{F}_k$. With this notation, $\mathbb{E}_0(X_1) = (Qf)(\xi_0) = \mathbb{E}(X_1|\xi_0)$. We denote by $\|X\|_p$ the norm in $\mathbb{L}_p = \mathbb{L}_p(\Omega, \mathcal{F}, \mathbb{P})$. The integral on the space $(S, \mathcal{A}, \pi)$ will be denoted by $\mathbb{E}_\pi$. So, $\mathbb{E}f(\xi_0) = \mathbb{E}_\pi f$.

The Markov chain is usually constructed in a canonical way on $\Omega = S^\infty$ endowed with sigma algebra $\mathcal{A}^\infty$, and ξ_n is the nth projection on S. The shift $T : \Omega \to \Omega$ is defined by $\xi_n(T\omega) = \xi_{n+1}(\omega)$ for every integer n.

For any probability measure υ on $\mathcal{A}$ the law of $(\xi_n)_{n\in\mathbb{Z}}$ with transition operator Q and initial distribution υ is the probability measure $\mathbb{P}^\upsilon$ on $(S^\infty, \mathcal{A}^\infty)$ such that

$$\mathbb{P}^\upsilon(\xi_{n+1} \in A|\xi_n = x) = Q(x, A) \quad \text{and} \quad \mathbb{P}^\upsilon(\xi_0 \in A) = \upsilon(A).$$

For $\upsilon = \pi$ we denote $\mathbb{P} = \mathbb{P}^\pi$. For $\upsilon = \delta_x$, the Dirac measure, we denote by $\mathbb{P}^x$ and $\mathbb{E}^x$ the probability and conditional expectation for the process started at x. Note that for each x fixed $\mathbb{P}^x(\cdot)$ is a measure on $\mathcal{F}^\infty$, the sigma algebra generated by $\cup_k \mathcal{F}_k$. Also

$$\mathbb{P}(A) = \int \mathbb{P}^x(A)\pi(dx). \tag{2}$$

We mention that any stationary sequence $(Y_k)_{k\in\mathbb{Z}}$ can be viewed as a function of a Markov process $\xi_k = (Y_j; j \leq k)$ with the function $g(\xi_k) = Y_k$. Therefore the theory of stationary processes can be embedded in the theory of Markov chains. So, our results apply to any stationary process with the corresponding interpretation. In the context of a stationary process, a fixed starting point for a corresponding Markov chain means a fixed past trajectory for $k \leq 0$.