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Affine realizations with affine state processes for stochastic partial differential equations

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Abstract

The goal of this paper is to clarify when a stochastic partial differential equation with an affine realization admits affine state processes. This includes a characterization of the set of initial points of the realization. Several examples, as the HJMM equation from mathematical finance, illustrate our results.

MSC: 60H15; 91G80

Keywords: Stochastic partial differential equation; Affine realization; Affine state process; Set of initial points

1. Introduction

The goal of this paper is to clarify when a semilinear stochastic partial differential equation (SPDE) of the form

$$\begin{cases} dr_t = (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t \\ r_0 = h_0 \end{cases} \tag{1.1}$$

in the spirit of [24] driven by a $\mathbb{R}^n$-valued Wiener process W (for some positive integer $n \in \mathbb{N}$) with an affine realization admits affine and admissible state processes. Affine realizations are particular types of finite dimensional realizations (FDRs). Denoting by H the state space of (1.1), which we assume to be a separable Hilbert space, the idea of a FDR is that for each starting point

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$h_0 \in \mathfrak{I}$ (where $\mathfrak{I} \subset H$ denotes the set of initial points) we can express the weak solution r to (1.1) locally as

$$r = \varphi(X) \tag{1.2}$$

for some $\mathbb{R}^d$-valued (typically time-inhomogeneous) process X and a deterministic mapping $\varphi : \mathbb{R}^d \to H$, which makes the infinite dimensional SPDE (1.1) more tractable. If we have a representation of the form (1.2), then the mapping φ is the parametrization of an invariant submanifold $\mathcal{M}$.

In this situation, the term *affine* has a twofold meaning, which we shall now explain. We speak about an affine realization if for each starting point $h_0 \in \mathfrak{I}$ we can express the weak solution r to (1.1) locally as

$$r = \psi + X \tag{1.3}$$

with a deterministic curve $\psi : \mathbb{T} \to H$, where $\mathbb{T} = [0, \delta]$ for some $\delta > 0$, and a process X having values in a state space of the form $\mathfrak{C} \oplus U$ with a finite dimensional proper cone $\mathfrak{C} \subset H$ and a finite dimensional subspace $U \subset H$. In this case, we also say that the SPDE (1.1) has an affine realization generated by $\mathfrak{C} \oplus U$, and the invariant manifold $(\mathcal{M}_t)_{t\in\mathbb{T}}$ is a collection of affine spaces

$$\mathcal{M}_t = \psi(t) + \mathfrak{C} \oplus U, \quad t \in \mathbb{T}, \tag{1.4}$$

also called a foliation, and the curve ψ is a parametrization of $(\mathcal{M}_t)_{t\in\mathbb{T}}$.

We say that such an affine realization has affine and admissible state processes if for each starting point $h_0 \in \mathfrak{I}$ the process X appearing in (1.3) is a (typically time-inhomogeneous) affine and admissible process on the state space $\mathfrak{C} \oplus U$. Here the term *affine* means that the local characteristics of X are affine, that is, the drift is affine and the volatility is square-affine, and the term *admissible* means that the state space $\mathfrak{C} \oplus U$ is invariant for X, which means that the drift is inward pointing and the volatility is parallel to the boundary at boundary points of $\mathfrak{C} \oplus U$.[1]

There is a substantial literature about FDRs for SPDEs, in particular for the HJMM equation from mathematical finance. Here we use the name HJMM equation, as it is the Heath–Jarrow–Morton (HJM) model from [20] with Musiela parametrization presented in [6]. The existence of FDRs for the HJMM equation driven by Wiener processes has intensively been studied in the literature, and we refer to [5,4,17,18] and references therein, and to [3] for a survey. As shown in [17], the existence of a FDR for the Wiener process driven HJMM equation implies the existence of an affine realization. The existence of affine realizations has been studied in [27] for the HJMM equation driven by Wiener processes, in [28,23] for the HJMM equation driven by Lévy processes, and in [29] for general SPDEs driven by Lévy processes.

Affine processes have found growing interest due to their analytical tractability, in particular regarding applications in the field of mathematical finance. We refer, e.g., to [11,12,10,13,16] for affine processes on the canonical state space, and, e.g., to [7,9,26] for affine processes on more general state spaces. We also mention the recent papers [2] and [8], where HJM-type models driven by affine processes are studied. Note that our state space $\mathfrak{C} \oplus U$ corresponds to the canonical state space $\mathbb{R}_+^m \times \mathbb{R}^{d-m}$.

The goal of this paper is to clarify when the SPDE (1.1) admits an affine realization with affine and admissible state processes – which has not been studied in the literature so far – and

[1] In the literature, a process is usually called an affine process if it is affine and admissible in the just described sense. For the purposes of this paper, we will carefully distinguish between the terms *affine* and *admissible*.