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Backward uniqueness of stochastic parabolic like equations driven by Gaussian multiplicative noise

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Abstract

One proves here the backward uniqueness of solutions to stochastic semilinear parabolic equations and also for the tamed Navier–Stokes equations driven by linearly multiplicative Gaussian noises. Applications to approximate controllability of nonlinear stochastic parabolic equations with initial controllers are given. The method of proof relies on the logarithmic convexity property known to hold for solutions to linear evolution equations in Hilbert spaces with self-adjoint principal part.

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1. Introduction

Consider the stochastic parabolic equation

$$\begin{aligned} &dX(t) - \sum_{i,j=1}^{d} \frac{\partial}{\partial \xi_i}\left(a_{ij}(t,\xi)\,\frac{\partial X(t)}{\partial \xi_j}\right)dt + b(t,\xi)\cdot\nabla X(t)\,dt \\ &\quad + \psi(t,\xi,X(t))dt = X(t)dW(t) \quad \text{in } (0,T)\times\mathcal{O}, \\ &X(0,\xi) = x(\xi), \quad \xi\in\mathcal{O}; \qquad X(t,\xi)=0 \quad \text{on } (0,T)\times\partial\mathcal{O}, \end{aligned} \tag{1.1}$$

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where $\mathcal{O} \subset \mathbb{R}^d,\ 1 \le d < \infty$, is a bounded and open domain with the smooth boundary $\partial\mathcal{O}$ and W is a Wiener process of the form

$$W(t,\xi) = \sum_{j=1}^{\infty} \mu_j e_j(\xi)\beta_j(t), \quad \xi \in \overline{\mathcal{O}},\ t \ge 0. \tag{1.2}$$

Here $\{e_j\}_{j=1}^{\infty} \subset C^2(\overline{\mathcal{O}})$ is an orthonormal basis in $L^2(\mathcal{O})$, $\{\beta_j\}_{j=1}^{\infty}$ is an independent system of real-valued Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $(\mathcal{F}_t)_{t\ge 0}$, and $\{\mu_j\}_{j=1}^{\infty} \subset \mathbb{R}$ is such that

$$\sum_{j=1}^{\infty} \mu_j^2 \|e_j\|^2_{C_b^2} < \infty, \tag{1.3}$$

where $\|\cdot\|_{C_b^2}$ denotes the supnorm of the functions and its first and second order derivatives.

As regards the functions $a_{ij} : [0,T]\times\overline{\mathcal{O}} \to \mathbb{R}$, $b : [0,T]\times\overline{\mathcal{O}} \to \mathbb{R}$ and $\psi : [0,T]\times\overline{\mathcal{O}} \to \mathbb{R}$, we assume that the following conditions hold

$$\begin{aligned} &a_{ij} \in C([0,T]\times\overline{\mathcal{O}}), \qquad \frac{\partial}{\partial t} a_{ij} \in C([0,T]\times\overline{\mathcal{O}}), \qquad \frac{\partial}{\partial \xi_j} a_{ij} \in C([0,T]\times\overline{\mathcal{O}}),\\ &a_{ij} = a_{ji}, \quad \forall i,j = 1,\dots,d, \qquad b \in C([0,T]\times\overline{\mathcal{O}};\mathbb{R}^d),\\ &\quad \mathrm{div}_\xi b \in C([0,T]\times\overline{\mathcal{O}}), \end{aligned} \tag{1.4}$$

$$\sum_{i,j=1}^{d} a_{ij}(t,\xi)u_i u_j \ge \gamma|u|^2, \quad \forall u = (u_1,\dots,u_d) \in \mathbb{R}^d,\ (t,\xi) \in [0,T]\times\overline{\mathcal{O}}, \tag{1.5}$$

where $\gamma > 0$ and $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$,

$$\psi \in C([0,T]\times\overline{\mathcal{O}}\times\mathbb{R}), \quad \psi(t,\xi,0) \equiv 0. \tag{1.6}$$

Moreover, $r \to \psi(t,\xi,r)$ is monotonically nondecreasing and

$$\begin{aligned} &|\psi(t,\xi,r_1) - \psi(t,\xi,r_2)| \le L|r_1 - r_2|\,|\psi_0(t,\xi,r_1,r_2)|,\\ &\quad \forall r_1, r_2 \in \mathbb{R},\ (t,\xi) \in [0,T]\times\overline{\mathcal{O}}, \end{aligned} \tag{1.7}$$

where $\psi_0 \in C([0,T]\times\overline{\mathcal{O}}\times\mathbb{R}\times\mathbb{R})$ and $L > 0$,

$$|\psi_0(t,\xi,r_1,r_2)| \le C(|r_1|^q + |r_2|^q + 1), \quad \forall r_1, r_2 \in \mathbb{R},\ (t,\xi) \in [0,T]\times\overline{\mathcal{O}}, \tag{1.8}$$

where

$$\begin{aligned} &0 \le q < \frac{4}{d-2} \quad \text{if } d > 2,\\ &q \in (0,\infty) \qquad \text{if } d = 2, \end{aligned} \tag{1.9}$$

and no polynomial growth condition of the form (1.9) is necessary if $d = 1$.

In the following, we denote by $L^2(\mathcal{O})$ the space of Lebesgue square integrable functions on $\mathcal{O}$ with the norm denoted by $|\cdot|_2$ and the scalar product $\langle\cdot,\cdot\rangle$. We denote by $W^{m,p}(\mathcal{O})$, $H_0^1(\mathcal{O})$ and $H^{-1}(\mathcal{O})$ the standard Sobolev spaces on $\mathcal{O}$ with the usual norms $\|u\|_{m,p}$, $\|\cdot\|_1$ and $\|\cdot\|_{-1}$, respectively.