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Stochastic Processes and their Applications 125 (2015) 2493–2515

stochastic processes and their applications

www.elsevier.com/locate/spa

# Optimal liquidity provision[☆]

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Received 18 September 2014; received in revised form 3 February 2015; accepted 25 February 2015
Available online 7 March 2015

**Abstract**

A small investor provides liquidity at the best bid and ask prices of a limit order market. For small spreads and frequent orders of other market participants, we explicitly determine the investor's optimal policy and welfare. In doing so, we allow for general dynamics of the mid price, the spread, and the order flow, as well as for arbitrary preferences of the liquidity provider under consideration.


*MSC:* 91B60; 91G10; 91G80

*Keywords:* Limit order markets; Optimal liquidity provision; Asymptotics

## 1. Introduction

Trades on financial markets are instigated by various motives. For example, mutual funds rebalance their portfolios, derivative positions are hedged, and margin calls may necessitate the liquidation of large asset positions. Such trades require counterparties who provide the necessary liquidity to the market. Traditionally, this market making role was played by designated

☆ The authors thank Pierre Collin-Dufresne, Paolo Guasoni, Jan Kallsen, Ren Liu, Mathieu Rosenbaum, and Torsten Schöneborn for fruitful discussions and two anonymous referees for valuable comments.
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http://dx.doi.org/10.1016/j.spa.2015.02.015

"specialists", who agreed on contractual terms to match incoming orders in exchange for earning the spread between their bid and ask prices. As stock markets have become automated, this quasi-monopolistic setup has given way to *limit order markets* on many trading venues. Here, electronic *limit order books* collect all incoming orders, and automatically pair matching buy and sell trades. Such limit order markets allow virtually all market participants to engage in systematic liquidity provision, which has become a popular algorithmic trading strategy for hedge funds.

The present study analyzes optimal strategies for liquidity provision and their performance. In contrast to most previous work on market making, we do *not* consider a single *large* monopolistic specialist (e.g., [10,2,15,3,12]) who optimally sets the bid–ask spread. Instead, as in [22,6,12,14,27], we focus on a *small* liquidity provider who chooses how much liquidity to provide by placing limit buy and sell orders at exogenously given bid and ask prices, respectively. For tractability, we assume that limit orders of the liquidity provider are fully executed against any incoming market order, and, by the above choice of the limit prices, her orders enjoy priority over limit orders submitted by other market participants. Thereby, we abstract from incentives to place orders at different limit prices, which leads to an enormous dimensionality reduction of the strategy space that has to be considered. To wit, we do not have to model the whole order book. Instead, our model is fully specified by the bid–ask price processes and the arrival times of market orders of other market participants. We assume that the mid-price of the risky asset follows a martingale and consider the practically relevant limiting regime of small spreads and frequent orders of other market participants. Thereby, we obtain explicit formulas in a general setting allowing for arbitrary dynamics of the mid price, the spread, and the order flow, as well as for general preferences of the liquidity provider under consideration.[1]

Given the liquidity provider's risk aversion, the asset's volatility, and the arrival rates of exogenous orders, the model tells us how much liquidity to provide by placing limit orders. However, our model abstracts from the precise microstructure of order books, in particular from the finite price grid and the use of information about order volumes in the book. In this spirit, we work with diffusion processes that are more tractable than integer-valued jump processes. Ignoring volume effects, our model carries the flavor of the standard frictionless market model and models with proportional transaction costs. Consequently, the model does not answer the question whether to place, say, the limit buy order of optimal size *exactly* at the current best bid price or possibly one tick above/below it.

In this setting, the optimal policy is determined by an upper and lower boundary for the monetary position in the risky asset, to which the investor trades whenever an exogenous market order of another market participant arrives. Hence, these target positions determine the amount of liquidity the investor posts in the limit order book. Kühn and Stroh [22] characterize these boundaries by the solution of a free boundary problem for a log-investor with unit risk aversion, who only keeps long positions in a market with constant order flow and bid–ask prices following geometric Brownian motion with positive drift. In the present study, we show in a general setting with a martingale mid price that – in the limit for small spreads and frequent orders of other market participants – the upper and lower target positions are given explicitly by

$$\overline{\beta}_t = \frac{2\varepsilon_t \alpha_t^{(2)}}{\mathrm{ARA}(x_0)\sigma_t^2}, \qquad \underline{\beta}_t = -\frac{2\varepsilon_t \alpha_t^{(1)}}{\mathrm{ARA}(x_0)\sigma_t^2}. \tag{1.1}$$

[1] Related results for models with small trading costs have recently been determined by [28,24,30,20,21]. These correspond to optimal trading strategies for liquidity takers, whose demand is matched by liquidity providers such as the ones considered here.