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Forward–backward stochastic differential systems associated to Navier–Stokes equations in the whole space[☆]

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Abstract

A coupled forward–backward stochastic differential system (FBSDS) is formulated in spaces of fields for the incompressible Navier–Stokes equation in the whole space. It is shown to have a unique local solution, and further if either the Reynolds number is small or the dimension of the forward stochastic differential equation is equal to two, it can be shown to have a unique global solution. These results are shown with probabilistic arguments to imply the known existence and uniqueness results for the Navier–Stokes equation, and thus provide probabilistic formulas to the latter. Related results and the maximum principle are also addressed for partial differential equations (PDEs) of Burgers' type. Moreover, from truncating the time interval of the above FBSDS, approximate solution is derived for the Navier–Stokes equation by a new class of FBSDSs and their associated PDEs; our probabilistic formula is also bridged to the probabilistic Lagrangian representations for the velocity field, given by Constantin and Iyer (2008) and Zhang

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(2010); finally, the solution of the Navier–Stokes equation is shown to be a critical point of controlled forward–backward stochastic differential equations.

Keywords: Forward–backward stochastic differential system; Navier–Stokes equation; Feynman–Kac formula; Strong solution; Lagrangian approach; Variational formulation

1. Introduction

Consider the following Cauchy problem for deterministic backward Navier–Stokes equation for the velocity field of an incompressible, viscous fluid:

$$\begin{cases} \partial_t u + \frac{\nu}{2}\Delta u + (u\cdot\nabla)u + \nabla p + f = 0, \quad t \le T; \\ \nabla\cdot u = 0, \quad u(T) = G, \end{cases} \tag{1.1}$$

which is obtained from the classical Navier–Stokes equation via the time-reversing transformation

$$(u, p, f)(t, x) \longmapsto (-u, p, f)(T-t, x), \quad \text{for } t \le T. \tag{1.2}$$

Here, $T > 0$, u is the d-dimensional velocity field of the fluid, p is the pressure field, $\nu \in (0,\infty)$ is the kinematic viscosity, $G = -u_0$ with u_0 being the initial velocity of the fluid by the above transformation, and f is the external force field which, without any loss of generality, is taken to be divergence free. It is well-known that the Navier–Stokes equation was introduced by Navier [37] and Stokes [50] via adding a dissipative term $\nu\Delta u$ as the friction force to Euler's equation, which is Newton's law for an infinitesimal volume element of the fluid.

Forward–backward stochastic differential equations (FBSDEs) are already well-known nowadays to be connected to systems of nonlinear parabolic partial differential equations (PDEs) (see among many others [2,10,17,30,35,41,42,51,57]). Within such a theory, the d-dimensional Burgers' equation (in the backward form)

$$\begin{cases} \partial_t v + \frac{\nu}{2}\Delta v + (v\cdot\nabla)v + f = 0, \quad t \le T; \\ v(T) = \phi, \end{cases} \tag{1.3}$$

as a simplified version of Navier–Stokes equation (1.1), is associated in a straightforward way to the following coupled FBSDE:

$$\begin{cases} dX_s(t,x) = Y_s(t,x)\,ds + \sqrt{\nu}\,dW_s, \quad s \in [t,T]; \\ X_t(t,x) = x; \\ -dY_s(t,x) = f(s, X_s(t,x))\,ds - \sqrt{\nu}Z_s(t,x)\,dW_s, \quad s \in [t,T]; \\ Y_T(t,x) = \phi(X_T(t,x)), \end{cases} \tag{1.4}$$

where W is a d-dimensional standard Brownian motion, X satisfies a *forward* stochastic differential equation and satisfies a backward one. They are related to each other by the following:

$$Y_s(t,x) = v(s, X_s(t,x)),\ Z_s(t,x) = \nabla v(s, X_s(t,x)), \quad s \in [t,T]\times\mathbb{R}^d \tag{1.5}$$