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Generalised particle filters with Gaussian mixtures

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Abstract

Stochastic filtering is defined as the estimation of a partially observed dynamical system. Approximating the solution of the filtering problem with Gaussian mixtures has been a very popular method since the 1970s. Despite nearly fifty years of development, the existing work is based on the success of the numerical implementation and is not theoretically justified. This paper fills this gap and contains a rigorous analysis of a new Gaussian mixture approximation to the solution of the filtering problem. We deduce the L^2-convergence rate for the approximating system and show some numerical examples to test the new algorithm.

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Keywords: Stochastic partial differential equation; Nonlinear filtering; Zakai equation; Particle filters; Gaussian mixtures; L^2-convergence

1. Introduction

The stochastic filtering problem deals with the estimation of an evolving dynamical system, called the *signal*, based on *partial observations* and a priori stochastic model. The signal is modelled by a stochastic process denoted by $X = \{X_t,\ t \geq 0\}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The signal process is not available to observe directly; instead, a partial observation is obtained and it is modelled by a process $Y = \{Y_t,\ t \geq 0\}$. The information available from the observation up to time t is defined as the filtration $\mathcal{Y} = \{\mathcal{Y}_t,\ t \geq 0\}$ generated by the ob-

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servation process Y. In this setting, we want to compute π_t — the conditional distribution of X_t given $\mathcal{Y}_t$. The analytical solutions to the filtering problem are rarely available, and there are only few exceptions such as the Kalman–Bucy filter and the Beneš filter (see, e.g. Chapter 6 in [4]). Therefore numerical algorithms for solving the filtering equations are required.

The description of a numerical approximation for π_t should contain the following three parts: the class of approximations; the law of evolution of the approximation; and the method of measuring the approximating error. Generalised particle filters with Gaussian mixtures is a numerical scheme to approximate the solution of the filtering problem, and it is a natural generalisation of the classic particle filters (or sequential Monte Carlo methods) in the sense that the Dirac delta measures used in the classic particle approximations are replaced by mixtures of Gaussian measures. In other words, Gaussian mixture approximations are algorithms that approximate π_t with random measures of the form

$$\sum_i a_i(t)\Gamma_{v_i(t),\omega_j(t)},$$

where $\Gamma_{v_j(t),\omega_j(t)}$ is the Gaussian measure with mean $v_j(t)$ and covariance matrix $\omega_j(t)$. The evolution of the weights, the mean and the covariance matrices satisfy certain stochastic differential equations which are numerically solvable. As time increases, typically the trajectories of a large number of particles diverge from the signal's trajectory; with only a small number remaining close to the signal. The weights of the diverging particles decrease rapidly, therefore contributing very little to the approximating system, and causing the approximation to converge very slowly to the conditional distribution. In order to tackle this so-called *sample degeneracy* phenomenon, a *correction procedure* is added. At correction times, each particle is replaced by a random number of offspring. Redundant particles are abandoned and only the particles contributing significantly to the system (i.e. with large weights) are carried forward; so that the most probable region of the trajectory of the signal process X will be more thoroughly explored. This correction mechanism is also called branching or resampling. Currently the multinomial branching algorithm and the tree based branching algorithm (TBBA) are two approaches for the correction step.

The idea of using Gaussian mixtures in the context of Bayesian estimation can be traced back to Alspach and Sorenson [1,32], and Tam and Moore [34]. Later in the work by Anderson and Moore [2], Gaussian mixtures are used in combination with extended Kalman filters to produce an empirical approximation to the discrete time filtering problem. The research on this area became more active since 2000s. Recently in Doucet et al. [15], these methods are revisited to construct a method which uses Rao-Blackwellisation in order to take advantage of the analytic structure present in some important classes of state-space models. Recent advances in this direction are contained in [3,14,11]. In Chen and Liu [6], a random mixture of Gaussian distributions, called mixture Kalman filters, are used to approximate a target distribution again based on the explicit linear filter. Further work on this topic is contained in [18,29,33,35]. Gustafsson et al. [20] describe a general framework for a number of applications, which are implemented using the idea of Gaussian particle filters. Further development in this direction can be found in [13,17,19,26]. For more recent work related to Gaussian mixture approximations, see Kotecha and Djurić [24,25], Le Gland et al. [27], Reich [30], Flament et al. [16], Van der Merwe and Wan [12], Carmi et al. [5], Iglesias, Law and Stuart [21], and Crisan and Li [7].

1.1. Contribution of the paper

In this paper we construct a new approximation to the conditional distribution of the signal $\pi = \{\pi_t : t \geq 0\}$ that consists of a mixture of Gaussian measures. In contrast with the exist-