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Stochastic Processes and their Applications 125 (2015) 2820–2855

stochastic processes and their applications

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Pathwise Taylor expansions for random fields on multiple dimensional paths

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Received 15 October 2013; received in revised form 7 February 2015; accepted 11 February 2015
Available online 20 February 2015

Abstract

In this paper we establish the pathwise Taylor expansions for random fields that are "regular" in terms of Dupire's path-derivatives [6]. Using the language of pathwise calculus, we carry out the Taylor expansion naturally to any order and for any dimension, which extends the result of Buckdahn et al. (2011). More importantly, the expansion can be both "forward" and "backward", and the remainder is estimated in a pathwise manner. This result will be the main building block for our new notion of viscosity solution to forward path-dependent PDEs corresponding to (forward) stochastic PDEs in our accompanying paper Buckdahn et al. [4].

MSC: 60H07, 15, 30; 35R60; 34F05

Keywords: Path derivatives; Pathwise Taylor expansion; Functional Itô formula; Itô–Wentzell formula; Stochastic partial differential equations

1. Introduction

In this paper we are interested in establishing the pathwise Taylor expansions for the Itô-type random field of the form

$$u(t,x) = u_0(x) + \int_0^t \alpha(s,x)ds + \int_0^t \beta(s,x) \circ dB_s, \tag{1.1}$$

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http://dx.doi.org/10.1016/j.spa.2015.02.004

where B is a d-dimensional standard Brownian motion, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and "$\circ$" means Stratonovich integral. In particular we are interested in such expansions for the solution to the following fully nonlinear stochastic partial differential equations (SPDE):

$$u(t,x) = u_0(x) + \int_0^t f(s,x,\cdot,u,\partial_x u,\partial_{xx}u)ds + \int_0^t g(s,x,\cdot,u,\partial_x u)\circ dB_s, \tag{1.2}$$

where f and g are random fields that have certain regularity in their spatial variables.

In our previous work [3] we studied the so-called *pathwise stochastic Taylor expansion* for a class of Itô-type random fields. The main result can be briefly described as follows. Suppose that u is a random field of the form (1.1), and B is a one dimensional Brownian motion. If we denote $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ to be the natural filtration generated by B and augmented by all $\mathbb{P}$-null sets in $\mathcal{F}$, then under reasonable regularity assumptions on the integrands α and β, the following stochastic "Taylor expansion" holds: For any stopping time τ and any $\mathcal{F}_\tau$-measurable, square-integrable random variable ξ, and for any sequence of random variables $\{(\tau_k, \xi_k)\}$ where τ_k's are stopping times such that either $\tau_k > \tau, \tau_k \downarrow \tau$; or $\tau_k < \tau,\ \tau_k \uparrow \tau$, and ξ_k's are all $\mathcal{F}_{\tau_k\wedge\tau}$-measurable, square integrable random variables, converging to ξ in L^2, it holds that

$$\begin{aligned} u(\tau_k,\xi_k) &= u(\tau,\xi) + a(\tau_k-\tau) + b(B_{\tau_k}-B_\tau) + p(\xi_k-\xi) + \frac{c}{2}(B_{\tau_k}-B_\tau)^2 \\ &\quad + q(\xi_k-\xi)(B_{\tau_k}-B_\tau) + \frac{1}{2}X(\xi_k-\xi)^2 + o(|\tau_k-\tau| + |\xi_k-\xi|^2), \end{aligned} \tag{1.3}$$

where (a,b,c,p,q,X) are all $\mathcal{F}_\tau$-measurable random variables, and the remainder $o(\zeta_k)$ are such that $o(\zeta_k)/\zeta_k \to 0$ as $k\to\infty$, in probability. Furthermore, the six-tuple (a,b,c,p,q,X) can be determined explicitly in terms of $\alpha,\ \beta$ and their derivatives (in certain sense).

While the Taylor expansion (1.3) reveals the possibility of estimating the remainder in a stronger form than mean-square (see for example [12]), it is not satisfactory for the study of pathwise property of the random fields which is essential in the study of, for example, stochastic viscosity solution. In a subsequent paper (Buckdahn–Bulla–Ma [2]) the result was extended to the case where the expansion could be made around any random time–space point (τ,ξ) where τ does not have to be a stopping time; and more importantly, the remainder was estimated in a pathwise manner, in the spirit of the Kolmogorov continuity criterion. In other words, modulo a $\mathbb{P}$-null set, the estimate holds for each ω, locally uniformly in (t,x). Furthermore, all the coefficients can be calculated explicitly in terms of a certain kind of "derivatives" for Itô-type random field, introduced in [2] (see more detailed description in Section 8 of this paper). It is noted, however, that a main drawback of the result in [2] is that the derivatives involved are not intuitive, and are difficult to verify. A more significant weakness of the result is that the dimension of the Brownian motion is restricted to 1, which, as we shall see in this paper, reduced the complexity of the Taylor expansion drastically.

The main purpose of this paper is to re-investigate the Taylor expansion in a much more general setting, but with a different "language". In particular, we shall allow both the spatial variable and the Brownian motion to be multi-dimensional, and the random field is "regular" in a very different way. To be more precise, we shall introduce a new notion of "path-derivative" in the spirit of Dupire [6] to impose a different type of regularity, that is, the regularity on the variable $\omega\in\Omega$. Such a language turns out to be very effective, and many originally cumbersome expressions in stochastic analysis becomes intuitive and very easy to understand. For example, the Itô–Wentzell formula reads exactly like the multi-dimensional Itô formula, and both inte-