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# Pathwise Itô calculus for rough paths and rough PDEs with path dependent coefficients

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**Abstract**

This paper introduces path derivatives, in the spirit of Dupire's functional Itô calculus, for controlled rough paths in rough path theory with possibly non-geometric rough paths. We next study rough PDEs with coefficients depending on the rough path itself, which corresponds to stochastic PDEs with random coefficients. Such coefficients are less regular in the time variable, which is not covered in the existing literature. The results are useful for studying viscosity solutions of stochastic PDEs.


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*Keywords:* Rough path; Functional Itô calculus; Path derivatives; Itô–Ventzell formula; Rough differential equations; Rough PDEs; Stochastic PDEs; Characteristics

## 1. Introduction

Firstly initiated by Lyons [33], rough path theory has been studied extensively and its applications have been found in many areas, including the recent application on KPZ equations by Hairer [24]. We refer to Lyons [34], Friz and Hairer [9], Friz and Victoir [21], and the references therein for the general theory and its applications.

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On the other hand, the functional Itô calculus, initiated by Dupire [13] and further developed by Cont and Fournie [9], has received very strong attention in recent years. In particular, it has proven to be a very convenient language for the theory of path dependent PDEs, see Peng and Wang [37], Ekren, Keller, Touzi and Zhang [14], and Ekren, Touzi and Zhang [15,16]. We also refer to Buckdahn, Ma and Zhang [6], Cosso and Russo [10], Leao, Ohashi and Simas [27], and Oberhauser [36] for some recent related works on functional Itô calculus.

The first goal of this paper is to develop the pathwise Itô calculus, in the spirit of Dupire's functional Itô calculus, in the rough path framework with possibly non-geometric rough paths. Based on the quadratic compensator of rough paths, which plays the role of quadratic variation in semimartingale theory, we introduce path derivatives for controlled rough paths of Gubinelli [22]. Our first order spatial path derivative is the same as Gubinelli's derivative, and the time derivative is closely related to a second order Taylor expansion of the controlled rough paths. This allows us to study the structure of a fairly general class of controlled rough paths, and more importantly, to treat rough path integration and rough ODEs/PDEs in the same manner as standard Itô calculus. In particular,

- the pathwise Taylor expansion and the pathwise Itô formula become equivalent;
- as observed by Buckdahn, Ma and Zhang [6] in a Brownian motion setting, the pathwise Itô–Ventzell formula is equivalent to the chain rule of our path derivatives, which is crucial for studying rough PDEs and stochastic PDEs;
- We may study rough ODEs/PDEs whose "drift term" is driven by the quadratic compensator, instead of $dt$. See (1.1) and (1.3). This is natural in semimartingale theory when the driving martingale is not a Brownian motion.

We shall remark though, while we believe such presentation of path derivatives in the rough path framework is new, many related ideas have already been discussed in the literature. Besides [18] and the reference therein, we also refer to the recent work Perkowski and Prömel [38] for some related studies.

We next study the following rough differential equations in the form:

$$d\theta_t = g(t,\theta_t)d\boldsymbol{\omega}_t + f(t,\theta_t)d\langle\boldsymbol{\omega}\rangle_t, \tag{1.1}$$

where $\boldsymbol{\omega}$ is a Hölder-$\alpha$ continuous rough path and $\langle\boldsymbol{\omega}\rangle$ is its quadratic compensator. We remark that, as mentioned in previous paragraph, we use Young's integration $f(t,\theta_t)d\langle\boldsymbol{\omega}\rangle_t$ rather than Lebesgue integration $f(t,\theta_t)dt$ in the "drift" term above, and they become the same when $\boldsymbol{\omega}$ is induced by a sample path of Brownian motion with Itô integration. Our study of above RDE is mainly motivated from the following stochastic differential equations with random coefficients:

$$dX_t = g(t,\omega,X_t)dB_t + f(t,\omega,X_t)dt, \tag{1.2}$$

where $B$ is a Brownian motion in the canonical probability space $(\Omega,\mathcal{F},\mathbb{P})$, $dB$ is Itô integration, and $g, f$ are adapted, namely depend on the history of the path: $\{\omega_s\}_{0\le s\le t}$. In the literature, typically the coefficients $g$ and $f$ in (1.1) do not depend on $t$, or at least is Hölder-$(1-\alpha)$ continuous in $t$, see Lejay and Victoir [28]. However, since a Brownian motion sample path $\omega$ is only Hölder-$(\frac{1}{2}-\varepsilon)$ continuous, by setting $\alpha = \frac{1}{2}-\varepsilon$, for (1.2) it is not reasonable to assume the mapping $t\mapsto g(\cdot,\omega,x)$ is Hölder-$(1-\alpha)$ continuous as required by [28]. Consequently, we are not able to apply the existing results in the rough path literature to study SDE (1.2) with random coefficients. We shall provide various estimates for rough path integrations, which follow more or less standard arguments, and then establish the well-posedness of RDE (1.1) under minimum regularity conditions on the coefficients. To be precise, we require only that $g(\cdot,x)$, $f(\cdot,x)$, and