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# Asymptotic theory for the sample covariance matrix of a heavy-tailed multivariate time series[☆]

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**Abstract**

In this paper we give an asymptotic theory for the eigenvalues of the sample covariance matrix of a multivariate time series. The time series constitutes a linear process across time and between components. The input noise of the linear process has regularly varying tails with index $\alpha \in (0, 4)$; in particular, the time series has infinite fourth moment. We derive the limiting behavior for the largest eigenvalues of the sample covariance matrix and show point process convergence of the normalized eigenvalues. The limiting process has an explicit form involving points of a Poisson process and eigenvalues of a non-negative definite matrix. Based on this convergence we derive limit theory for a host of other continuous functionals of the eigenvalues, including the joint convergence of the largest eigenvalues, the joint convergence of the largest eigenvalue and the trace of the sample covariance matrix, and the ratio of the largest eigenvalue to their sum.


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*Keywords:* Regular variation; Sample covariance matrix; Dependent entries; Largest eigenvalues; Trace; Point process convergence; Compound Poisson limit; Infinite variance stable limit; Fréchet distribution

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## 1. Introduction

In the setting of classical multivariate statistics or multivariate time series, the data consist of $n$ observations of $p$-dimensional random vectors, where $p$ is relatively small compared to the sample size $n$. With the recent advent of large data sets, the dimension $p$ can be large relative to the sample size and hence standard asymptotics, assuming $p$ is fixed relative to $n$ may provide misleading results. Structure in multivariate data is often summarized by the sample covariance matrix. For example, principal component analysis, extracts principal component vectors corresponding to the largest eigenvalues. Consequently, there is a need to study asymptotics of the largest eigenvalues of the sample covariance matrix. In the case of $p$ fixed and the $p \times n$ data matrix consists of i.i.d. $N(0, 1)$ observations, Anderson [1] derived the limit law for the largest eigenvalue. In a now seminal paper, Johnstone [11] showed that if $p_n \to \infty$ at the rate $p_n/n \to \gamma \in (0, \infty)$, then the largest eigenvalues, suitable normalized, converges to the *Tracy–Widom* distribution with $\beta = 1$. Johnstone's result has been generalized by Tao and Vu [19] where only 4 moments are needed to determine the limit. The theory for the largest eigenvalues of sample covariance and Wigner matrices based on heavy tails is not as well developed as in the light-tailed case. The largest eigenvalues of sample covariance matrices with i.i.d. entries that are regularly varying with index $-\alpha$ were studied by Soshnikov [18] for the $\alpha \in (0, 2)$ case and subsequently extended in Auffinger et al. [2] to the $\alpha \in (2, 4)$ case. They showed that the point process of eigenvalues, normalized by the square of the $1-(np)^{-1}$ quantile converges in distribution to a Poisson point process with intensity $(\alpha/2)x^{-\alpha/2-1}$, provided $p/n \to \gamma$, where $\gamma \in (0, 1)$. These results were extended in Davis et al. [6] to the case where the rows of the data matrix are i.i.d. linear heavy-tailed processes. They also had more general growth conditions on $p_n$ in the case of i.i.d. entries and $\alpha \in (0, 2)$.

In this paper, we study the asymptotic behavior of the largest eigenvalues of the sample covariance matrices of a multivariate time series. The time series is assumed to be heavy-tailed and linearly dependent in time and between the components. Even though [6] allowed for some dependence between the rows, it was somewhat contrived in that the rows were assumed to be conditionally independent given a random process. To our knowledge, the present paper is the first to consider bona fide dependence among the components in the time series which renders a multivariate analysis, such as PCA, meaningful. Allowing for dependence between the rows can appreciably impact the limiting behavior of the largest eigenvalues. Instead of obtaining a Poisson point process as the limit of the extreme eigenvalues, we now get a "cluster" Poisson point process. That is, the limit can be described by a Poisson point process in which each point produces a "cluster" of points. The clusters are determined via the eigenvalues of an auxiliary *covariance matrix* that is constructed from the linear filter weights. Interestingly, the limit point process is identical to the limit point process derived by Davis and Resnick [7] for the extremes of a linear process. One of the striking differences in the limit theory between the independent and dependent row cases is the limiting behavior of the ratio of the second largest eigenvalue, $\lambda_{(2)}$ to the largest eigenvalue $\lambda_{(1)}$ of the sample covariance matrix. In the independent row case,

$$\lambda_{(2)}/\lambda_{(1)} \xrightarrow{d} U^{\alpha/2},$$

where $U$ is a uniform random variable on $(0, 1)$ and $\alpha \in (0, 2)$ is the index of regular variation. Now if the rows are dependent, then the limit random variable corresponds to a truncated uniform, i.e., there exists a constant $c \in [0, 1)$ such that the limit has the form $c^{\alpha/2} I_{\{U<c\}} + U^{\alpha/2} I_{\{U \geq c\}}$. The constant $c$ is determined from the eigenvalues of the auxiliary covariance matrix.