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Weak approximation of martingale representations

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Abstract

We present a systematic method for computing explicit approximations to martingale representations for a large class of Brownian functionals. The approximations are obtained by computing a directional derivative of the weak Euler scheme and yield a consistent estimator for the integrand in the martingale representation formula for any square-integrable functional of the solution of an SDE with path-dependent coefficients. Explicit convergence rates are derived for functionals which are Lipschitz-continuous in the supremum norm. Our results require neither the Markov property, nor any differentiability conditions on the functional or the coefficients of the stochastic differential equations involved.

Keywords: Martingale representation; Semimartingale; Functional calculus; Functional Ito calculus; Clark–Ocone formula; Malliavin calculus; Stochastic differential equations; Euler approximation

0. Introduction

Let W be a standard d-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t)$ its ($\mathbb{P}$-completed) natural filtration. Then, for any square-integrable $\mathcal{F}_T$-measurable random variable H, or equivalently, any square-integrable $(\mathcal{F}_t)$-martingale $Y(t) = \mathbb{E}[H|\mathcal{F}_t]$, there exists a unique $(\mathcal{F}_t)$-predictable process ϕ with $\mathbb{E}\left[\int_0^T \operatorname{tr}(\phi(u)^t \phi(u))du\right] < \infty$ such that:

$$H = Y(T) = \mathbb{E}[H] + \int_0^T \phi \cdot dW. \tag{1}$$

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The classical proof of this representation result (see e.g. [33]) is non-constructive. However in many applications, such as stochastic control or mathematical finance, one is interested in an explicit expression for ϕ, which represents an optimal control or a hedging strategy. Expressions for the integrand ϕ have been derived using a variety of methods and assumptions, using Markovian techniques [10,13,15,23,30], integration by parts [2] or Malliavin calculus [1,3,18, 21,24,27,28]. Some of these methods are limited to the case where Y is a Markov process; others require differentiability of H in the Fréchet or Malliavin sense [3,17,18,28] or an explicit form for the density [2]. Almost all of these methods invariably involve an approximation step, either through the solution of an auxiliary partial differential equation (PDE) or the simulation of an auxiliary stochastic differential equation.

A systematic approach to obtaining martingale representation formulae has been proposed in [5], using the Functional Itô calculus [6,7,12]: it was shown in [5, Theorem 5.9] that for any square-integrable $(\mathcal{F}_t)$-martingale Y,

$$\forall t \in [0,T], \quad Y(t) = Y(0) + \int_0^t \nabla_W Y \cdot dW \quad \mathbb{P}\text{−a.s.}$$

where $\nabla_W Y$ is the weak vertical derivative of Y with respect to W, constructed as an L^2 limit of pathwise directional derivatives. This approach does not rely on any Markov property nor on the Gaussian structure of the Wiener space and is applicable to functionals of a large class of Itô processes.

In the present work we build on this approach to propose a general framework for computing explicit approximations to the integrand ϕ in a general setting in which H is allowed to be a functional of the solution of a stochastic differential equation (SDE) with path-dependent coefficients:

$$dX(t) = b(t, X_t)dt + \sigma(t, X_t)dW(t) \qquad X(0) = x_0 \in \mathbb{R}^d \tag{2}$$

where $X_t = X(t \wedge .)$ designates the path stopped at t and

$$b : [0,T] \times D([0,T], \mathbb{R}^d) \to \mathbb{R}^d, \qquad \sigma : [0,T] \times D([0,T], \mathbb{R}^d) \to \mathbb{M}_d(\mathbb{R})$$

are continuous non-anticipative functionals. For any square-integrable variable of the form $H = g(X(t), 0 \le t \le T)$ where $g : (D([0,T], \mathbb{R}^d), \|.\|_\infty) \to \mathbb{R}$ is a continuous functional, we construct an explicit sequence of approximations ϕ_n for the integrand ϕ in (1). These approximations are constructed as vertical derivatives, in the sense of the functional Itô calculus[5], of the weak Euler approximation F_n of the martingale Y, obtained by replacing X by the corresponding Euler scheme ${}_nX$:

$$\phi_n(t) = \nabla_\omega F_n(t, W), \quad \text{where } F_n(t,\omega) = \mathbb{E}\left[g({}_nX(\omega \underset{t}{\oplus} W)) \right]$$

where $\underset{t}{\oplus}$ is the concatenation of paths at t and $\nabla_\omega F_n$ is the Dupire derivative [4,12], a directional derivative defined as a pathwise limit of finite-difference approximations. and thus readily computable path-by-path in a simulation setting.

The main results of the paper are the following. We first show the existence and continuity of these pathwise derivatives in Theorem 3.1. The convergence of the approximations ϕ_n to the integrand ϕ in (1) is shown in Proposition 4.1. Under a Lipschitz assumption on g, we provide in Theorem 4.1 an L^p error estimate for the approximation error. The proposed approximations are easy to compute and readily integrated in commonly used numerical schemes for SDEs.