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Excursion probability of certain non-centered smooth Gaussian random fields

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Abstract

Let $X = \{X(t), t \in T\}$ be a non-centered, unit-variance, smooth Gaussian random field indexed on some parameter space T, and let $A_u(X, T) = \{t \in T : X(t) \geq u\}$ be the excursion set. It is shown that, as $u \to \infty$, the excursion probability $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$ can be approximated by the expected Euler characteristic of $A_u(X, T)$, denoted by $\mathbb{E}\{\chi(A_u(X, T))\}$, such that the error is super-exponentially small. The explicit formulae for $\mathbb{E}\{\chi(A_u(X, T))\}$ are also derived for two cases: (i) T is a rectangle and $X - \mathbb{E}X$ is stationary; (ii) T is an N-dimensional sphere and $X - \mathbb{E}X$ is isotropic.

MSC: 60G15; 60G60; 60G70

Keywords: Excursion probability; Gaussian random fields; Euler characteristic; Rectangle; Sphere; Super-exponentially small

1. Introduction

Let $X = \{X(t),\ t \in T\}$ be a real-valued Gaussian random field living on some parameter space T. The excursion probability $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$ has been extensively studied in the literature due to its importance in both theory and applications in many areas. We refer to the survey Adler [1] and monographs Piterbarg [11], Adler and Taylor [2] and Azaïs and

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Wschebor [5] for the history, recent developments and related applications on this subject. To approximate the excursion probability for high exceeding level u, many authors have developed various powerful tools, including the double sum method [11], the tube method [15], the expected Euler characteristic approximation [1,16,17,2] and the Rice method [3–5].

In particular, the expected Euler characteristic approximation establishes a very general and profound result, building an interesting connection between the excursion probability and the geometry of the field. It was first rigorously proved by Taylor et al. [17] (see also Theorem 14.3.3 in [2]), showing that for a centered, unit-variance, smooth Gaussian random field, under certain conditions on the regularity of X and topology of T,

$$\mathbb{P}\left\{\sup_{t\in T} X(t) \geq u\right\} = \mathbb{E}\{\chi(A_u(X,T))\}\big(1 + o\big(e^{-\alpha u^2}\big)\big) \quad \text{as } u \to \infty, \tag{1.1}$$

where $\chi(A_u(X,T))$ is the Euler characteristic of the excursion set $A_u(X,T) = \{t \in T : X(t) \geq u\}$ and $\alpha > 0$ is some constant. This verifies the "Expected Euler Characteristic Heuristic" for centered, unit-variance, smooth Gaussian random fields. Similar results can be found in [5] where the Rice method was applied. It had also been further developed by Cheng and Xiao [8] that (1.1) holds for certain Gaussian fields with stationary increments which have nonconstant variances. However, to the best of our knowledge, among the existing works on deriving the expected Euler characteristic approximation (1.1), the Gaussian field X is always assumed to be centered. In fact, the study of excursion probability for non-centered Gaussian fields is also very valuable since the varying mean function plays an important role in many models. Especially, when the Gaussian field is non-smooth, several results on the excursion probability have been obtained via the double sum method (see, for examples, [11,13,10]).

In this paper, we study the excursion probability $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$ for non-centered, unit variance, smooth (see condition (H1) below) Gaussian random fields. As the first contribution, we obtain in Theorem 3.5 that, in general, the expected Euler characteristic approximation 2.5 holds for such non-centered Gaussian fields when $T \subset \mathbb{R}^N$ is a compact rectangle. It shows that, comparing with the double sum method for non-smooth non-centered Gaussian fields (see [13] for example), we are able to obtain a much more accurate approximation for the excursion probability of smooth non-centered Gaussian fields such that the error is super-exponentially small. This is because the expected Euler characteristic approximation takes into account the effect of X over the boundary of T, which is ignored in the double sum method. By similar arguments in [3], such approximation can also be easily extended to the cases when $T \subset \mathbb{R}^N$ is a compact and convex set with smooth boundary or a compact and smooth manifold without boundary, see Theorem 2.6.

To apply the approximation in practice, one needs to find an explicit formula for the expected Euler characteristic $\mathbb{E}\{\chi(A_u(X,T))\}$. Under the assumption of centered Gaussian fields, Taylor and Adler [16] showed a very nice formula for $\mathbb{E}\{\chi(A_u(X,T))\}$ (see also [2]), involving the Lipschitz–Killing curvatures of the excursion set $A_u(X,T)$. However, there is lack of research to evaluate $\mathbb{E}\{\chi(A_u(X,T))\}$ for non-centered Gaussian fields. We provide here explicit formulae of $\mathbb{E}\{\chi(A_u(X,T))\}$ for two cases of non-centered Gaussian fields: (i) T is a rectangle and $X - \mathbb{E}X$ is stationary; (ii) T is an N-dimensional sphere and $X - \mathbb{E}X$ is isotropic; see respectively Theorems 3.5 and 3.11. The results show that, the mean function of the field does make the formula of $\mathbb{E}\{\chi(A_u(X,T))\}$ much more complicated than that of the centered field. In real applications, one usually needs to use the Laplace method to obtain explicit asymptotics for $\mathbb{E}\{\chi(A_u(X,T))\}$.