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Fluctuation theorems for synchronization of interacting Pólya's urns

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Abstract

We consider a system of N two-colors urns in which the reinforcement of each urn depends also on the content of all the other urns. This interaction is of mean-field type and it is tuned by a parameter $\alpha \in [0, 1]$; in particular, for $\alpha = 0$ the N urns behave as N independent Pólya's urns. For $\alpha > 0$ urns synchronize, in the sense that the fraction of balls of a given color converges a.s. to the same (random) limit in all urns. In this paper we study fluctuations around this synchronized regime. The scaling of these fluctuations depends on the parameter α. In particular the standard scaling $t^{-1/2}$ appears only for $\alpha > 1/2$. For $\alpha \geq 1/2$ we also determine the limit distribution of the rescaled fluctuations. We use the notion of stable convergence, which is stronger than convergence in distribution.

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1. Introduction

In this paper we continue the study of synchronization for a model of interacting Pólya's urn that has been introduced in [9]. This study is motivated by the attempt of understanding the role of reinforcement in synchronization phenomena. Here the word *synchronization* is meant in the wide sense of "coherent behavior of the majority", that could be time stationary or time periodic. Experimental results in the context of cellular and neuronal systems have stimulated the formulation and the analysis of stylized stochastic models that could reveal the origin of such phenomena, in particular in systems that are not time-reversible (see e.g. [13,25]). The wide majority of the models proposed consist of time-homogeneous Markov processes; with this choice, long-time correlations and *aging* are usually ruled out.

One way of breaking time-homogeneity and adding memory to the dynamics consists in introducing a reinforcement mechanism. The most popular stylized model in this context is the Pólya's urn model. In the simplest version, the model consists of an urn which contains balls of two different colors (for example, at time $t = 0$, $a \geq 1$ red and $b \geq 1$ black balls). At each discrete time a ball is drawn out and it is replaced in the urn together with another ball of the same color. Let Z_t be the fraction of red balls at time t, namely, the conditional probability of drawing a red ball at time $t + 1$, given the fraction of the red balls at time t. A well known result (see for instance [17] or [20]) states that $(Z_t)_t$ is a bounded martingale and in particular $\lim_{t\to\infty} Z_t = Z_\infty$ a.s., where Z_∞ has Beta distribution with parameters a and b.

In [9] an interacting version of this model is formulated. Consider a set of $N > 1$ Pólya's urns and introduce a "mean field interaction" among them:

- at time 0, each urn contains $a \geq 1$ red and $b \geq 1$ black balls;
- at each time $t + 1$, a new ball is introduced in each urn and, given the fraction $Z_t(i)$, for $1 \leq i \leq N$, of red balls in each urn i at time t, the ball added in urn j is, independently of what happens for all the other urns, red (otherwise it is black) with conditional probability $\alpha Z_t + (1 - \alpha)Z_t(j)$, where Z_t is the total fraction of red balls in the system at time t, i.e. $Z_t = \frac{1}{N}\sum_{i=1}^{N} Z_t(i)$, and $\alpha \in [0, 1]$.

The case $\alpha = 0$ corresponds to N independent copies of the classical Pólya's urn described above. Thus, for $1 \leq j \leq N$, each proportion $Z_t(j)$ converges, as $t \to +\infty$, to i.i.d. random variables, whose distribution is Beta with parameters a, b. As soon as $\alpha > 0$, some basic properties of the Pólya's urn model are lost: in particular the sequences $\big(Z_t(j)\big)_t$ are not martingales (although $(Z_t)_t$ is a martingale), and the sequences of the colors drawn are no more exchangeable. It is shown in [9], that $D_t(j) := Z_t(j) - Z_t$ converges to zero almost surely and in L^2; as a consequence, all the fractions $Z_t(j)$ converge a.s. to the *same limit* Z. We refer to this phenomenon as *almost sure synchronization* of the system of interacting urns. It is relevant to note that this is not a macroscopic or *thermodynamic* effect: the number of urns N is kept fixed. The "phase transition" from disorder ($\alpha = 0$) to synchronization ($\alpha > 0$) is not a consequence of the large scale of the system but of the long memory caused by the reinforcement.

In the present paper we analyze in detail the *fluctuations* of $D_t(j) = Z_t(j) - Z_t$ around zero as $t \to +\infty$. The rate of convergence to zero in L^2-norm has been already analyzed in [9], revealing an interesting scaling for certain values of the interaction parameter α: $D_t(j)$ scales as $t^{-1/2}$ for $\frac{1}{2} < \alpha \leq 1$, as $t^{-1/2}\sqrt{\ln(t)}$ for $\alpha = \frac{1}{2}$, and as $t^{-\alpha}$ for $0 < \alpha < \frac{1}{2}$. In this paper, we obtain limit theorems for the rescaled fluctuations: for $\alpha \geq \frac{1}{2}$, they converge in distribution, as $t \to +\infty$, to a mixture of centered Gaussian distributions, whose random variance is an explicit function of the limit random variable Z; for $0 < \alpha < \frac{1}{2}$, the rescaled fluctuations converge *almost*