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On the almost sure topological limits of collections of local empirical processes at many different scales

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Abstract

Let h_n and $\mathfrak{h}_n$ be two bandwidth sequences both pertaining to the domain of the strong local invariance principle, but tending to zero at different rates. We investigate the almost sure uniform clustering of Strassen type for collections of local (or increments of) empirical processes at a fixed point, under localizing scales $h \in [h_n, \mathfrak{h}_n]$. We show that, within the framework of Strassen functional limit laws for local empirical processes, and whenever $\log\log(\mathfrak{h}_n/h_n)/\log\log(n) \to \delta > 0$, the collections of all increments along bandwidths $h \in [h_n, \mathfrak{h}_n]$ almost surely admit an inner and outer topological limit. Those are Strassen balls with respective radii $\sqrt{\delta}$ and $\sqrt{1+\delta}$.

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1. Introduction and statement of main results

Denote by $(U_i)_{i\geq 1}$ an independent, identically distributed (i.i.d.) sequence of random variables having the uniform distribution on $[0, 1]$. Due to their wide range of statistical applications, the empirical and quantile processes have been subject of deep studies since half

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a century. Among those was established the Strassen limit law for the functional increments of the uniform empirical processes [13]. The latter can be briefly mentioned as follows, writing $\log_2(n) := \log(\log(n \vee e))$: given a deterministic sequence $h_n \in (0, 1]$ fulfilling

$$h_n \downarrow 0,\ nh_n \uparrow,\ nh_n/\log_2(n) \to \infty, \tag{1.1}$$

and given $t_0 \in [0, 1)$, the normalized functional increments of the uniform empirical process (at t_0)

$$\Delta\alpha_n(\cdot, h_n, t_0) := \frac{\sqrt{n}\Big(\mathbb{F}_n(t_0 + h_n\cdot) - \mathbb{F}_n(t_0) - h_n\cdot\Big)}{\sqrt{2h_n \log_2(n)}}, \quad \text{with}$$

$$\mathbb{F}_n(t) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{[0,t]}(U_i), \quad t \in [0, 1],$$

almost surely admits $\mathcal{S}$ as a cluster set in $\ell^\infty([0, 1])$. Here $\mathcal{S}$ denotes the Strassen set, i.e. the unit ball of the reproducing kernel Hilbert space of the Wiener process on [0, 1]. One can equivalently say that the random sequence of singletons $\{\Delta\alpha_n(\cdot, h_n, t_0)\}$ almost surely admits $\emptyset$ and $\mathcal{S}$ as inner and outer topological limits (see, e.g., [12, p. 122]). In this article, we partially provide an answer to the following question:

> "If one takes two sequences $h_n \leq \mathfrak{h}_n$ fulfilling (1.1), but tending to zero at different rates, what is the limiting behavior of the sequence of collections of random increments $\big\{\Delta\alpha_n(\cdot, h, t_0),\ h \in [h_n, \mathfrak{h}_n]\big\}$?"

Such a problem is – at least at first sight – non trivial because of the heuristic that, if $h_n/\mathfrak{h}_n \to 0$ then $\Delta\alpha_n(\cdot, h_n, t_0)$ and $\Delta\alpha_n(\cdot, \mathfrak{h}_n, t_0)$ are asymptotically independent – see [3,18] for concrete forms of that heuristic. Hence indexing by $h \in [h_n, \mathfrak{h}_n]$ requires to control increasing collections of "almost mutually independent" local empirical processes.

We provide an answer to this problem when the sequence $\log_2(\mathfrak{h}_n/h_n)/\log_2(n)$ admits a non null limit δ, and we derive explicit inner and outer topological limits equal to $\sqrt{\delta}\mathcal{S}$ and $\sqrt{1+\delta}\mathcal{S}$ respectively. In addition (see Section 1.3) we shall illustrate the statistical implications of such a result to the uniform in bandwidth strong consistency of estimators—here, estimators of Hill type for the tail index. However our main result will not be limited to the $\Delta\alpha_n(\cdot, h, t_0)$. Indeed, in two breakthrough articles, Einmahl and Mason [10] and then Mason [14] showed that a large panel of limit laws of Strassen or Donsker type holding for the $\Delta\alpha_n(\cdot, h, t_0)$ could be extended to more general mathematical objects pertaining to the theory of empirical processes indexed by functions—or abstract empirical process. They showed that the usual *uniform entropy* and *bracketing* conditions play the same significant role as for usual abstract empirical processes (see, e.g., [17]). Our work naturally takes places in that much wider framework, more precisely the local empirical processes as defined by Mason [14]. The next two pages essentially consist in a reminder of that framework. To that end, we shall make use of the classical notations in empirical processes theory, that are employed in [17].

1.1. A class of local empirical processes

Given an (i.i.d.) sequence $(Z_i)_{i\geq 1}$ from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to $\mathbb{R}^d$, a point $z \in \mathbb{R}^d$, a real number $h > 0$ – usually called a bandwidth – and a Borel measurable real function g on