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On the dual problem of utility maximization in incomplete markets

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Abstract

In this paper, we study the dual problem of the expected utility maximization in incomplete markets with bounded random endowment. We start with the problem formulated in Cvitanić et al. (2001) and prove the following statement: in the Brownian framework, the countably additive part $\widehat{Q}^r$ of the dual optimizer $\widehat{Q} \in (L^\infty)^*$ obtained in Cvitanić et al. (2001) can be represented by the terminal value of a supermartingale deflator Y defined in Kramkov and Schachermayer (1999), which is a local martingale.

MSC: 91B28; 91B16

Keywords: Utility maximization; Random endowment; Primal–dual approach; Dual optimizer

1. Introduction

Optimal investment is a classical problem in mathematical finance, which concerns an economic agent who invests in a financial market so as to maximize the expected utility of his terminal wealth. In this paper, we consider the utility maximization problem in general semimartingale markets and focus on the primal–dual approach, precisely, we are interested in properties of the solution to the dual problem.

A complete review of the literature on optimal investment is too extensive, so we only concentrate on those of immediate interest. In the semimartingale framework, Kramkov and

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Schachermayer study the problem with a utility function U supported on $\mathbb{R}_+$ in [19]. They assume the agent is endowed with a deterministic initial wealth x and can trade using admissible strategies—those which keep the corresponding wealth process bounded from below. The core approach in [19] is to construct an optimal trading strategy by solving a dual problem defined on L^0. The result in [19] is subsequently generalized by Cvitanić et al. [5] to allow for receiving a bounded random endowment e_T other than the deterministic initial wealth, which, for example, can be an exogenous non-traded European contingent claim. In this case, the primal problem under consideration is formulated as

$$u(x) := \sup_{g\in\mathcal{C}} \mathbf{E}[U(x+g+e_T)], \quad x \in \mathbb{R},$$

where $\mathcal{C}$ is the convex cone of all random variables dominated by admissible stochastic integrals. The authors of [5] employ the duality between L^∞ and $(L^\infty)^*$ and solve a minimization dual problem over the subset $\mathcal{D}$ of $(L^\infty)^*$, which can be regarded as the weak-star closure of the set $\mathcal{M}$ of equivalent local martingale measures (ELMMs). Precisely, the dual problem is formulated as

$$v(y) := \inf_{Q\in\mathcal{D}} \left\{ \mathbf{E}\left[V\left(y\frac{dQ^r}{d\mathbf{P}} \right) \right] + y\langle Q, e_T\rangle \right\}, \quad y > 0, \tag{1.1}$$

where V is the conjugate of U, $\mathbf{P}$ is the physical probability measure, and Q^r is the regular part of $Q \in (L^\infty)^*$. It states in [5] that a dual optimizer $\widehat{Q}$ can be found in $\mathcal{D}$, which is unique up to the singular part, and moreover the primal optimizer can be formulated in terms of $\widehat{Q}^r$.

In [15], the authors relax the boundedness assumption on the random endowment and instead only require an integrability condition. Another extension of [5] is provided by Karatzas and Žitković in [17] by allowing for intertemporal consumption.

Schachermayer treats the case of utility functions supporting both positive and negative wealth for a locally bounded risky asset without random endowment in [30]. Within this locally bounded semimartingale framework, Owen obtains a generalized result in [26] with a bounded random endowment and furthermore, Owen and Žitković [27] consider unbounded claims. It is worth mentioning that with the settings of [30], the dual optimizer is always a martingale measure but may not be equivalent, stated for example in [1,26,30]. On the other hand, Biagini and Frittelli investigate a problem similar to [30] and allow for a general semimartingale model for the risky asset. Their utility maximization problem is established on a new domain in [2] and is afterwards embedded in Orlicz spaces in [3]. Eventually, Biagini et al. [4] study the problem with unbounded random endowment by generalizing the duality method in [3].

Besides the duality between the primal and dual problem, many authors are concerned with the properties of the dual optimizer in the case when U is defined on $\mathbb{R}_+$; particularly, the following representation conjecture is studied: *the dual optimizer can be attained by an equivalent local martingale deflator* (*for short ELMDs, see e.g.* [16,18]). When $e_T = 0$, Kramkov and Schachermayer observe that this conjecture fails for general semimartingale models, since there is an example showing that the associated process could only be a strict supermartingale (cf. Example 5.1' in [19]). However, once further conditions are imposed on S, one could have some positive results. For example, Karatzas and Žitković observe that this conjecture is true for Itô-process models in [17], and in a recent paper of Horst et al. [14], this dual optimal process is constructed via solutions of backward stochastic differential equations. Another important result is presented in [23]: the representation conjecture is true for all continuous semimartingale models.

In the present paper, we study the regular part of the dual optimizer to (1.1) and establish the following main result: