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Multi-dimensional backward stochastic differential equations of diagonally quadratic generators☆

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Abstract

In this paper, we study a multi-dimensional BSDE with a "diagonally" quadratic generator, the quadratic part of whose ith component depends only on the ith row of the second unknown variable. Local and global solutions are given, which seem to be the first systematic (positive) results on the general solvability of multi-dimensional quadratic BSDEs. In our proofs, it is natural and crucial to apply both John–Nirenberg and reverse Hölder inequalities for BMO martingales. Our results are finally illustrated to solve the system of "diagonally" quadratic BSDEs arising from a nonzero-sum risk-sensitive stochastic differential game, which answers the open problem posed in El Karoui and Hamadène [Stochastic Process. Appl. 107 (2003), page 164].

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Keywords: Backward stochastic differential equation; Quadratic BSDE; BMO martingale; John–Nirenberg inequality; The reverse Hölder inequality

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1. Introduction

Consider the following backward stochastic differential equation (BSDE):

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s, \quad t \in [0, T]. \tag{1.1}$$

Here, $\{W_t := (W_t^1, \ldots, W_t^d)^*, 0 \le t \le T\}$ is a d-dimensional standard Brownian motion defined on some probability space $(\Omega, \mathscr{F}, P)$. Denote by $\{\mathscr{F}_t, 0 \le t \le T\}$ the augmented natural filtration of the standard Brownian motion W. The function $g : \Omega \times [0, T] \times R^n \times R^{n\times d} \to R^n$ is called the generator of BSDE (1.1). The history of BSDEs (1.1) can be dated back to Bismut in [1] for the linear case, and in [2] for a specifically structured matrix-valued nonlinear case where the matrix-valued generator contains a quadratic form of the second unknown. The uniformly Lipschitz case was later studied by Pardoux and Peng in [14].

Bismut [2] derived a matrix-valued BSDE of a *quadratic* generator—the so-called backward stochastic Riccati equation (BSRE) in the study of linear quadratic optimal control with random coefficients, while he could not solve it in general (see also Peng [15,16] for his exposition and comments on BSREs). In that paper, Bismut described the difficulty and failure of his fixed point techniques in the proof of the existence and uniqueness for BSDE of a quadratic generator (i.e., the so-called quadratic BSDE). Bismut's paper [2] has inspired subsequent intensive efforts in the research of quadratic BSDE (1.1). Nowadays numerous progress has been made in this issue: Kobylanski [11] and Briand and Hu [3,4] gave the existence and uniqueness result for the case of a scalar-valued ($n = 1$) quadratic BSDE, Tang [17,18] solved (using the stochastic maximum principle in [17] and dynamic programming in [18]) the existence and uniqueness result (posed by Bismut [2]) for the general BSRE, and Tevzadze [19] proved the existence and uniqueness result for multi-dimensional quadratic BSDE (1.1) under the assumption that the terminal value is *sufficiently small* in the supremum norm (also called the *small terminal value problem*). Frei and dos Reis [7] constructed a counterexample to show that multi-dimensional quadratic BSDE (1.1) might fail to have a global solution (Y, Z) on $[0, T]$ such that Y is essentially bounded, which illustrates the difficulty of the quadratic part contributing to the underlying scalar generator as an unbounded process—the exponential of whose time-integral is likely to have no finite expectation. Very recently, Cheridito and Nam [5] addressed a special system of quadratic BSDEs in the Markovian context, which arises from the equilibrium problem with interacting agents in a financial market (see Frei and dos Reis [7] for further descriptions). We also note the following two recent studies on systems of BSDEs with small terminal condition. Kramkov and Pulido [12] proved, under small terminal conditions, that a particular system of fully coupled quadratic BSDEs, coming from a price impact model, is globally well-posed, and provided a counterexample with bounded terminal condition where the system is not well-posed. Kramkov and Pulido [13] studied stability and asymptotic behavior of solutions for a general class of multi-dimensional quadratic BSDEs, again under small terminal conditions. Neither global nor local (in time) positive results are found in the literature for a general solvability of multidimensional quadratic BSDEs.

As mentioned above, the systems of quadratic BSDEs arise naturally in many different situations, and the existing results are only obtained under small terminal conditions. And there exists some system of quadratic BSDEs which does not admit any bounded solution. So it is interesting, from the theoretical point of view, to find some sufficient condition on the generator, in which case the corresponding system of quadratic BSDEs admits a unique local or global solution.