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Large-maturity regimes of the Heston forward smile

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Abstract

We provide a full characterisation of the large-maturity forward implied volatility smile in the Heston model. Although the leading decay is provided by a fairly classical large deviations behaviour, the algebraic expansion providing the higher-order terms highly depends on the parameters, and different powers of the maturity come into play. As a by-product of the analysis we provide new implied volatility asymptotics, both in the forward case and in the spot case, as well as extended SVI-type formulae. The proofs are based on extensions and refinements of sharp large deviations theory, in particular in cases where standard convexity arguments fail.

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1. Introduction

Consider an asset price process $\left(\mathrm{e}^{X_t}\right)_{t\geq 0}$ with $X_0 = 0$, paying no dividend, defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with a given risk-neutral measure $\mathbb{P}$, and assume that interest rates are zero. In the Black–Scholes–Merton (BSM) model, the dynamics of the logarithm of the asset price are given by $\mathrm{d}X_t = -\frac{1}{2}\sigma^2\mathrm{d}t + \sigma\mathrm{d}W_t$, where $\sigma > 0$ represents the instantaneous volatility and W is a standard Brownian motion. The no-arbitrage price of the call option at time zero is then given by the famous BSM formula [14,52]:

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$C_{\mathrm{BS}}(\tau, k, \sigma) := \mathbb{E}\left(\mathrm{e}^{X_\tau} - \mathrm{e}^k\right)_+ = \mathcal{N}(d_+) - \mathrm{e}^k \mathcal{N}(d_-)$, with $d_\pm := -\frac{k}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}$, where $\mathcal{N}$ is the standard normal distribution function. For a given market price $C^{\mathrm{obs}}(\tau, k)$ of the option at strike e^k and maturity τ, the spot implied volatility $\sigma_\tau(k)$ is the unique solution to the equation $C^{\mathrm{obs}}(\tau, k) = C_{\mathrm{BS}}(\tau, k, \sigma_\tau(k))$.

For any $t, \tau > 0$ and $k \in \mathbb{R}$, we define as in [13,51] a forward-start option with forward-start date t, maturity τ and strike e^k as a European option with payoff $\left(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^k\right)^+$ where $X_\tau^{(t)} := X_{t+\tau} - X_t$ pathwise. By the stationary increment property, its value is simply $C_{\mathrm{BS}}(\tau, k, \sigma)$ in the BSM model. For a given market price $C^{\mathrm{obs}}(t, \tau, k)$ of the option at strike e^k, forward-start date t and maturity τ, the forward implied volatility smile $\sigma_{t,\tau}(k)$ is then defined (see also [13]) as the unique solution to $C^{\mathrm{obs}}(t, \tau, k) = C_{\mathrm{BS}}(\tau, k, \sigma_{t,\tau}(k))$. The forward smile is a generalisation of the spot implied volatility smile, and the two are equal when $t = 0$.

The literature on implied volatility asymptotics is extensive and has drawn upon a wide range of mathematical techniques. Small-maturity asymptotics have received wide attention using heat kernel expansion results [8]. More recently, they have been studied using PDE methods [12,38,55], large deviations [21,23], saddlepoint methods [25], Malliavin calculus [9,48] and differential geometry [32,39]. Roger Lee [50] was the first to study extreme strike asymptotics, and further works on this have been carried out by Benaim and Friz [6,7] and in [35–37,28,21,18]. Large-maturity asymptotics have only been studied in [59,24,40,41,26] using large deviations and saddlepoint methods. Fouque et al. [27] have also successfully introduced perturbation techniques in order to study slow and fast mean-reverting stochastic volatility models. Models with jumps (including Lévy processes), studied in the above references for large maturities and extreme strikes, 'explode' in small time, in a precise sense investigated in [1,2,58,54,53,22].

On the other hand the literature on asymptotics of forward-start options and the forward smile is sparse. Glasserman and Wu [34] use different notions of forward volatilities to assess their predictive values in determining future option prices and future implied volatility. Keller-Ressel [47] studies the forward smile asymptotic when the forward-start date t becomes large (τ fixed). Bompis [15] produces an expansion for the forward smile in local volatility models with bounded diffusion coefficient. In [43] the authors compute small and large-maturity asymptotics for the forward smile in a general class of models (including stochastic volatility and time-changed exponential Lévy models) where the forward characteristic function satisfies certain properties (in particular essential smoothness of the re-scaled limit). In [42] the authors prove that for fixed $t > 0$ the Heston forward smile explodes as τ tends to zero. Finally, empirical results on the forward smile have been carried out by practitioners in Balland [5], Bergomi [13], Bühler [16] and Gatheral [31].

Under some conditions on the parameters, it was shown in [43] that the smooth behaviour of the pointwise limit $\lim_{\tau\uparrow\infty} \tau^{-1} \log \mathbb{E}(\mathrm{e}^{uX_\tau^{(t)}})$ yielded an asymptotic behaviour for the forward smile as $\sigma_{t,\tau}^2(k\tau) = v_0^\infty(k) + v_1^\infty(k, t)\tau^{-1} + \mathcal{O}(\tau^{-2})$, where $v_0^\infty(\cdot)$ and $v_1^\infty(\cdot, t)$ are continuous functions on $\mathbb{R}$. In particular for $t = 0$ (spot smiles), they recovered the result in [24] (also under some restrictions on the parameters). Interestingly, the limiting large-maturity forward smile v_0^∞ does not depend on the forward-start date t. A number of practitioners (see e.g. Balland [5]) have made the natural conjecture that the large-maturity forward smile should be the same as the large-maturity spot smile. The result above rigorously shows us that this indeed holds if and only if the Heston correlation is close enough to zero.

It is natural to ask what happens when the parameter restrictions are violated. We identify a number of regimes depending on the correlation and derive asymptotics in each regime. The