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Some effects of the noise intensity upon non-linear stochastic heat equations on [0, 1]

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Abstract

Various effects of the noise intensity upon the solution $u(t, x)$ of the stochastic heat equation with Dirichlet boundary conditions on [0, 1] are investigated. We show that for small noise intensity, the pth moment of $\sup_{x\in[0,1]} |u(t, x)|$ is exponentially stable, however, for large one, it grows at least exponentially. We also prove that the noise excitation of the pth energy of $u(t, x)$ is 4, as the noise intensity goes to infinity. We formulate a common method to investigate the lower bounds of the above two different behaviors for large noise intensity, which are hard parts in Foondun and Joseph (2014), Foondun and Nualart (2015) and Khoshnevisan and Kim (2015).

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1. Introduction and main results

We are interested in various behaviors of the following stochastic heat equation relative to λ:

$$\begin{cases} \partial_t u(t,x) &= \dfrac{1}{2}\Delta u(t,x) + \lambda\sigma(u(t,x))\dot{w}(t,x), \quad t>0,\ x\in(0,1), \\ u(0,x) &= u_0(x), \quad x\in(0,1), \end{cases} \tag{1.1}$$

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where $\lambda > 0$ is a positive number, σ is a non-random measurable function defined on $\mathbb{R}$ and $\dot{w}(t, x)$ is a Gaussian space–time noise on $[0, \infty) \times [0, 1]$, which is usually explained as the distribution derivative of Brownian sheet $w(t, x)$ in t and x, see [22]. Such an equation is closely connected to the parabolic Anderson model (as $\sigma(u) = u$, see [3]), the stochastic Burger's equation [1,13] and the Kardar–Parisi–Zhang (KPZ) equation [1,11,14]. Hence some crucial properties, such as the weak intermittency of the solution, are actively studied, see [4,9,17] and references therein.

In this paper, we are mainly interested in (1.1) with homogeneous Dirichlet boundary conditions, i.e., $u(t, 0) = u(t, 1) = 0$. Parts of our results will also hold for (1.1) with homogeneous Neumann boundary conditions $\partial_x u(t, 0) = \partial_x u(t, 1) = 0$ and we will state them in the form of remarks.

According to [10,16], the parameter $\lambda > 0$ in (1.1) will be called the level of noise or noise intensity, which is regarded as the inverse temperature. The solution $u(t, x)$ can be thought of as the partition function of a continuous space–time random polymer, see [2] for more explanations.

In this paper, two kinds of the behaviors of the solution relative to noise intensity λ will be studied. To explain our aims and motivations in detail, let us first introduce some notation and the definition of the solution (1.1). Let $\{\mathcal{F}_t\}_{t\geq 0}$ denote the filtration generated by the Brownian sheet $\{w(t, x); t \geq 0, x \in [0, 1]\}$, see [22] for more information. In this paper, we will always assume that the following assumptions hold:

(A.1) The initial value u_0 is non-random and continuous on $[0, 1]$. Furthermore, we assume that the Lebesgue measure of the set $[\gamma, 1-\gamma] \subset supp(u_0)$ is strictly positive, and $\inf_{x\in[\gamma,1-\gamma]} u_0(x) > 0$, where $supp(u_0)$ denotes the support of u_0 and $\gamma \in (0, 1/4)$ is fixed hereafter.

(A.2) $\sigma(0) = 0$ and σ is Lipschitz continuous, that is, there exists $K_U > 0$ such that for all $u, v \in \mathbb{R}$,

$$|\sigma(u) - \sigma(v)| \leq K_U |u - v|.$$

Let us recall the definition of the solution to (1.1). Based on the definition introduced in [22], a random field $\{u_\lambda(t, x); t \geq 0, x \in [0, 1]\}$ is said to be a mild solution of (1.1) with the homogeneous Dirichlet boundary conditions if it is $\mathcal{F}_t$-adapted and continuous in (t, x), and further it satisfies the following integral equation with probability one:

$$\begin{aligned} u(t, x) &= \int_0^1 g_D(t, x, y) u_0(y) dy + \int_0^t \int_0^1 g_D(t - s, x, y) \lambda \sigma(u(s, y)) w(dsdy) \\ &:= D_1(t, x) + D_{2,\lambda}(t, x), \end{aligned} \tag{1.2}$$

where $g_D(t, x, y)$ denotes the fundamental solution (or heat kernel) of $\partial_t u(t, x) = \frac{1}{2}\Delta u(t, x)$ with Dirichlet boundary conditions $u(t, 0) = u(t, 1) = 0$.

Similarly, an $\mathcal{F}_t$-adapted and continuous random field $\{u_\lambda(t, x); t \geq 0, x \in [0, 1]\}$ is said to be a mild solution of (1.1) with homogeneous Neumann boundary conditions if (1.2) is satisfied almost surely replaced $g_D(t, x, y)$ by the Neumann kernel $g_N(t, x, y)$, please see Chapter 3 [22] for its precise meaning. In addition, for the introduction to stochastic partial differential equations, we also refer the reader to [5] for more information.

Since our topics are closely dependent on the noise intensity λ, we will denote by $u_\lambda(t, x)$ the solution of (1.1) with homogeneous Dirichlet boundary conditions. Let $p \geq 2$ in this paper and then for any real valued measurable function u defined on $[0, 1]$, let $\|u\|_{L^p}$ denote its L^p-norm on $[0, 1]$. In particular, let us recall that for $p = \infty$, $\|u\|_{L^\infty} = \text{ess sup}_{x\in[0,1]} |u(x)|$.