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Minimal thinness with respect to subordinate killed Brownian motions

Panki Kim[a,*], Renming Song[b], Zoran Vondraček[c,b]

[a] Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Building 27, 1 Gwanak-ro, Gwanak-gu, Seoul 151-747, Republic of Korea

[b] Department of Mathematics, University of Illinois, Urbana, IL 61801, USA

[c] Department of Mathematics, University of Zagreb, Zagreb, Croatia

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Abstract

Minimal thinness is a notion that describes the smallness of a set at a boundary point. In this paper, we provide tests for minimal thinness for a large class of subordinate killed Brownian motions in bounded $C^{1,1}$ domains, $C^{1,1}$ domains with compact complements and domains above graphs of bounded $C^{1,1}$ functions.
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1. Introduction

Let $X = (X_t, \mathbb{P}_x)$ be a Hunt process in an open set $D \subset \mathbb{R}^d$, $d \geq 2$. Let $\partial_M D$ and $\partial_m D$ be the Martin and minimal Martin boundary of D with respect to X respectively. For any $z \in \partial_M D$, we denote by $M^D(x, z)$ the Martin kernel of D at z with respect to X. The family of all excessive functions for X will be denoted by $\mathcal{S}$. For a function $v : D \to [0, \infty]$ and a set $E \subset D$, the

* Corresponding author.
E-mail addresses: pkim@snu.ac.kr (P. Kim), rsong@math.uiuc.edu (R. Song), vondra@math.hr (Z. Vondraček).

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reduced function of v on E is defined by $R_v^E = \inf\{s \in \mathcal{S} : s \geq v \text{ on } E\}$ and its lower semi-continuous regularization is denoted by $\widehat{R}_v^E$. A set $E \subset D$ is said to be *minimally thin* in D at $z \in \partial_m D$ with respect to X if $\widehat{R}^E_{M^D(\cdot,z)} \neq M^D(\cdot, z)$, cf. [13]. A probabilistic interpretation of minimal thinness is given in terms of the process X conditioned to die at $z \in \partial_m D$: For any $z \in \partial_m D$, let $X^z = (X_t^z, \mathbb{P}_x^z)$ denote the $M^D(\cdot, z)$-process, Doob's h-transform of X with $h(\cdot) = M^D(\cdot, z)$. The lifetime of X^z will be denoted by ζ. It is known (see [23]) that $\lim_{t\uparrow\zeta} X_t^z = z$, $\mathbb{P}_x^z$-a.s. For $E \subset D$, let $T_E := \inf\{t > 0 : X_t^z \in E\}$. It is proved in [13, Satz 2.6] that a set $E \subset D$ is minimally thin at $z \in \partial_m D$ with respect to X if and only if there exists $x \in D$ such that $\mathbb{P}_x^z(T_E < \zeta) \neq 1$. This shows that minimal thinness is a concept describing smallness of a set at a boundary point.

The history of minimal thinness goes back to Lelong-Ferrand [24] who introduced this concept in case of the half-space in the setting of classical potential theory. Minimal thinness for general open sets was developed in Naïm [26], while probabilistic interpretation (in terms of Brownian motion) was given by Doob (see e.g. [11]). Various versions of Wiener-type criteria for minimal thinness were developed over the years culminating in the work of Aikawa [1] who, by using the powerful concept of quasi-additivity of capacity, established a criterion for minimal thinness for subsets of NTA domains. For a good exposition of these results and methods cf. [3, Part II,7]. In case of a $C^{1,1}$ domain $D \subset \mathbb{R}^d$, the finite part of the minimal Martin boundary $\partial_m D$ coincides with the Euclidean boundary ∂D, and Aikawa's criterion reads as follows: Let E be a Borel subset of D. If E is minimally thin at $z \in \partial D$, then

$$\int_{E\cap B(z,1)} |x - z|^{-d}\, dx < \infty. \tag{1.1}$$

Conversely, if E is the union of a subfamily of Whitney cubes of D and (1.1) holds, then E is minimally thin in D at z.

Note that all works listed above pertain to the classical potential theory related to Brownian motion. For more general Hunt processes, although the general theory of minimal thinness was developed by Föllmer already in 1969, see [13], until recently no concrete criteria for minimal thinness were known. The first paper addressing this question was [19] which dealt with minimal thinness of subsets of the half-space for a large class of subordinate Brownian motions. Quite general results for a large class of symmetric Lévy processes in κ-fat open sets were obtained in [22]. The special case of a $C^{1,1}$ open set D was given in [22, Corollary 1.5]. We present here a slightly simplified version of the main result of [22]. Assume that X is an isotropic Lévy process in $\mathbb{R}^d$, $d \geq 2$, with characteristic exponent $\Psi(x) = \Psi(|x|)$ satisfying the following weak scaling condition: There exist constants $0 < \delta_1 \leq \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1\lambda^{2\delta_1}\Psi(t) \leq \Psi(\lambda t) \leq a_2\lambda^{2\delta_2}\Psi(t), \quad \lambda \geq 1, t \geq 1. \tag{1.2}$$

We note that many subordinate Brownian motions, particularly all isotropic stable processes, satisfy the above condition. Let X^D be the process X killed upon exiting a $C^{1,1}$ open set D. If a Borel set $E \subset D$ is minimally thin in D at $z \in \partial D$ with respect to X^D, then (1.1) holds true. The converse is also true provided E is the union of a subfamily of Whitney cubes of D. Thus one obtains the same Aikawa-type criterion for minimal thinness regardless of the particular isotropic Lévy process X as long as X satisfies the weak scaling condition (1.2). This is a somewhat surprising result. An explanation for this hinges on sharp two-sided estimates for the Green function of X^D which imply that the singularity of the Martin kernel $M^D(x, z)$ near $z \in \partial D$ is of the order $|x - z|^{-d}$ for all such processes.