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Thick points for a Gaussian Free Field in 4 dimensions

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Abstract

This article is concerned with the study of fractal properties of thick points for a 4-dimensional Gaussian Free Field. We adopt the definition of Gaussian Free Field on $\mathbb{R}^4$ introduced by Chen and Jakobson (2012) viewed as an abstract Wiener space with underlying Hilbert space $H^2(\mathbb{R}^4)$. We can prove that for $0 \le a \le 4$, the Hausdorff dimension of the set of a-high points is $4 - a$. We also show that the thick points give full mass to the Liouville Quantum Gravity measure on $\mathbb{R}^4$.

MSC: 60G60; 60G15

Keywords: KPZ; Liouville quantum gravity; Thick points; Hausdorff dimension; Abstract Wiener space; Bilaplacian

1. Introduction

Random measures defined by means of log-correlated Gaussian fields X and that can be formally written as "$m(\mathrm{d}\omega) = e^{aX(\omega)}\mathrm{d}\omega$" arise in conformal field theory and in the theory of Gaussian multiplicative Chaos (GMC). When X is an instance of the Gaussian Free Field (GFF) these measures are referred to as Liouville quantum gravity (LQG) measures. The interest around such objects comes from physics and in particular from the understanding and proving the KPZ relation, formulated by Knizhnik, Polyakov and Zamolodchikov [17], which gives the relation between volume exponents derived using the quantum metric induced by $m(\mathrm{d}\omega)$ and the

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Euclidean metric. Several interesting papers have been written to show this relation, and the first result is given by Duplantier and Sheffield [10] proving the formula for the planar case. For a clear explanation of this and other aspects of the KPZ we refer to Garban [11]. To construct such measures one has to rely on an approximation (cut-off) of the field and there are various methods to construct this approximation. While on the one hand a more geometric approach (which explicitly relies on the structure of the field) is present in the work Duplantier and Sheffield [10], the perspective of Robert and Vargas [26,27,24] dates back to the definition of Mandelbrot [21], Kahane [16] of multiplicative chaos, which deals with properties of the covariance kernel. These works extended the concept of multiplicative chaos of Kahane to a more general class of covariance kernels.

In this paper we focus our attention on the *multifractal formalism* of the underpinned Gaussian field, or with an equivalent terminology on its so-called *thick points*. To our knowledge the first rigorous study in this direction was made by Mandelbrot [22] in the context of one-dimensional log-correlated Gaussian fields. Hu et al. [13] showed that the Hausdorff dimension of the set of a-thick points is $2 - a$ for $0 \le a \le 2$ for the planar GFF (case of sphere average process). In Kahane [16], Rhodes and Vargas [24] such a result is shown for certain covariance kernels leading to multiplicative chaos. In this article we extend the results of Hu et al. [13] to 4 dimensions using the sphere average process introduced by Chen and Jakobson [3]. The set of thick points is relevant in understanding the support of the LQG. In fact it was shown in Duplantier and Sheffield [10] that the LQG measure is almost surely supported on the thick points, in analogy to Kahane's similar results [16] on 1D Gaussian multiplicative chaos and to Rhodes and Vargas [24, Theorem 4.1] in higher dimensions.

To give an analogy in $\mathbb{Z}^d$, one might look at the discrete Gaussian free field. It undergoes a phase transition at $d = 2$ in terms of the existence of an infinite-volume limit measure. Similarly the discrete membrane model (whose covariance is the inverse of the discrete Bilaplacian) shows the same change of phase in $d = 4$ (further results about it can be found for instance in Kurt [18,20]). In the critical dimension both fields possess logarithmically growing variances, and moreover the results contained in Daviaud [5] and Cipriani [4] show a similar fractal behavior of the thick points. In the continuum case, a natural analogue of the membrane model would be the Gaussian field arising from the inverse (in the sense of distributions) of the Bilaplacian operator. However, it is still an open problem to derive for it an appropriate sphere average in the sense of Duplantier and Sheffield [10]. In this direction, Chen and Jakobson [3] first constructed the sphere average process for the massive Bilaplacian Gaussian free field.

The construction of the set of thick points relies on the choice of cut-offs. One of them is the sphere average process $X_\epsilon(x)$, which can be taken as the average of the field over a ball of radius ϵ around x (in the rest of the paper we will assume the parameters denoted by ϵ, ϵ_1 etc. to be small). The main advantage of such cut-offs is that they enjoy the spatial Markov property, that is, informally, the processes $(X_{t+s}(x) - X_s(x))_{t\ge 0}$ and $(X_{t+s}(y) - X_s(y))_{t\ge 0}$ are independent whenever $\|x - y\|$ is large enough. Cut-offs can also be created by truncating appropriately the covariance function [24], or using the orthonormal basis representation for generalized Gaussian fields [14]. We prefer to stick to the more geometrical construction of the Gaussian free field, as in Chen and Jakobson [3] rather than handling it as an instance of multiplicative chaos, in the framework of Rhodes and Vargas [24, Theorem 4.2] although both approaches prove to be fruitful to investigate thick points. Differences between the two approaches are discussed in Section 2.1.

Main results and structure of the article: In Section 2 we recall the model introduced by Chen and Jakobson [3] and state our main result more precisely. We show in Theorem 2.1 that the set of