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Particle systems with a singular mean-field self-excitation. Application to neuronal networks

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Abstract

We discuss the construction and approximation of solutions to a nonlinear McKean–Vlasov equation driven by a singular self-excitatory interaction of the mean-field type. Such an equation is intended to describe an infinite population of neurons which interact with one another. Each time a proportion of neurons 'spike', the whole network instantaneously receives an excitatory kick. The instantaneous nature of the excitation makes the system singular and prevents the application of standard results from the literature. Making use of the Skorohod M1 topology, we prove that, for the right notion of a 'physical' solution, the nonlinear equation can be approximated either by a finite particle system or by a delayed equation. As a by-product, we obtain the existence of 'synchronized' solutions, for which a macroscopic proportion of neurons may spike at the same time.

Keywords: McKean nonlinear diffusion process; Counting process; Propagation of chaos; Integrate-and-fire network; Skorohod M1 topology; Neuroscience

1. Introduction

Recently several rigorous studies [3–5,7] have been concerned with a mean-field equation modeling the behavior of a very large (infinite) network of interacting spiking neurons proposed in [14] (see also [1,8,10,12] and references therein for other types of mean-field models mo-

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tivated by neuroscience). As a nonlinear SDE in one-dimension the equation for the electrical potential X_t across any typical neuron in the network at time t takes the form

$$X_t = X_0 + \int_0^t b(X_s)ds + \alpha\mathbb{E}(M_t) + W_t - M_t, \quad t \geqslant 0, \tag{1.1}$$

where $X_0 < 1$ almost surely, $(W_t)_{t\geqslant 0}$ is a standard Brownian motion and b is a Lipschitz function of linear growth. Here α is a parameter in (0, 1) and the process $M = (M_t)_{t\geqslant 0}$ counts the number of times that $X = (X_t)_{t\geqslant 0}$ reaches 1 before time t, so that it is integer-valued (see Section 2 for a precise description). The idea is that when X reaches the threshold 1, M instantly increases by 1 so that X is reset to a value below the threshold, and we say that the neuron has *spiked*. Throughout the article we will write $e(t) := \mathbb{E}(M_t)$.

Eq. (1.1) is in fact nontrivial, since the form of the nonlinearity is not regular enough for the application of the standard McKean–Vlasov theory [13,17]. Indeed, the problem is that, on the infinitesimal level, the mean-field term in (1.1) reads as $e'(t) = [d/dt]\mathbb{E}(M_t)$, which is by no means regular with respect to the law of X_t. In [7], it is proven that $e'(t) = -(1/2)\partial_y p(t, 1)$, where $p(t, y)dy = \mathbb{P}(X_t \in dy)$ is the marginal density of X_t, which shows how singular the dependence of $e'(t)$ upon the law of X_t is. As such, most of the previous work studying this equation has been focused on the existence of a solution and its properties, bringing to light some nontrivial mechanisms.

The main point is that, for some choices of parameters (α too big for fixed X_0 concentrated close to the boundary), any solution to (1.1) must exhibit what has been described as a 'blow-up' in finite time. More precisely this means that $e'(t)$ (which is the mean-firing rate of the network at time t) must become infinite for some finite t. This was done in [3] by means of a PDE method. Interpreting (1.1) as a description of an infinite network of neurons, a blow-up is thus a time at which a proportion of all the neurons in the network spike at exactly the same time, which we refer to as a synchronization. Despite the interest in this phenomena, up until now it has been unclear how to continue a solution after a blow-up. On the other hand, in [7] it was shown by probabilistic arguments that for other choices of parameters (α small enough for fixed $X_0 = x_0$), (1.1) has a unique solution for all time which does not exhibit the blow-up phenomenon. These two complementary results are made precise in Theorems 2.3 and 2.4.

The aim of the present work is to provide further insight into this nonlinear equation by providing two ways of approximating (and moreover constructing) a solution. The first is via the natural particle system associated to (1.1), which describes the behavior of the finite network of neurons. In fact, the introduction of (1.1) in [14] is inspired from this finite dimensional system: it is there asserted that, when the size of the network becomes infinite, neurons become independent and evolve according to (1.1). However, the proof of this fact (which is a propagation of chaos result) is not given. The first of our main objectives is to fill this gap and to rigorously show that any weak limit of the particle system must be a solution to (1.1) (see Theorem 4.4). In particular, we show that the particle system converges to the solution of (1.1) whenever uniqueness holds, in which case propagation of chaos holds as well. Again, due to the irregularity and nature of the particle system, this result is in fact more difficult than it might appear. The second objective is to recover a similar result when approximating the self-interaction in (1.1) by delayed self-interactions (see Theorem 4.6). The motivation for considering the delayed equation (which is still nonlinear) is that it never exhibits a blow-up phenomenon, even with α close to 1, making it easier to handle (see Proposition 3.5).

In both cases, the strategy relies on two ingredients. First, we show that there exists a notion of 'physical' solutions to Eq. (1.1) for which spikes occur physically, in a 'sequential' way. The