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Holderian weak invariance principle under a Hannan type condition

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Abstract

We investigate the invariance principle in Hölder spaces for strictly stationary martingale difference sequences. In particular, we show that the sufficient condition on the tail in the i.i.d. case does not extend to stationary ergodic martingale differences. We provide a sufficient condition on the conditional variance which guarantee the invariance principle in Hölder spaces. We then deduce a condition in the spirit of Hannan one.

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1. Introduction

One of the main problems in probability theory is the understanding of the asymptotic behavior of Birkhoff sums $S_n(f) := \sum_{j=0}^{n-1} f \circ T^i$, where $(\Omega, \mathcal{F}, \mu, T)$ is a dynamical system and f a map from Ω to the real line.

One can consider random functions constructed from the Birkhoff sums

$$S_n^{\mathrm{pl}}(f, t) := S_{[nt]}(f) + (nt - [nt]) f \circ T^{[nt]+1}, \quad t \in [0, 1] \tag{1.1}$$

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and investigate the asymptotic behavior of the sequence $\left(S_n^{\mathrm{pl}}(f,t)\right)_{n\geq 1}$ seen as an element of a function space. Donsker showed (cf. [4]) that the sequence $(n^{-1/2}(\mathbb{E}(f^2))^{-1/2}S_n^{\mathrm{pl}}(f))_{n\geq 1}$ converges in distribution in the space of continuous functions on the unit interval to a standard Brownian motion W when the sequence $(f\circ T^i)_{i\geq 0}$ is i.i.d. and zero mean. Then an intensive research has then been performed to extend this result to stationary weakly dependent sequences. We refer the reader to [9] for the main theorems in this direction.

Our purpose is to investigate the weak convergence of the sequence $(n^{-1/2}S_n^{\mathrm{pl}}(f))_{n\geq 1}$ in Hölder spaces when $(f\circ T^i)_{i\geq 0}$ is a strictly stationary sequence. A classical method for showing a limit theorem is to use a martingale approximation, which allows to deduce the corresponding result if it holds for martingale difference sequences provided that the approximation is good enough. To the best of our knowledge, no result about the invariance principle in Hölder space for stationary martingale difference sequences is known.

1.1. The Hölder spaces

It is well known that standard Brownian motion's paths are almost surely Hölder regular of exponent α for each $\alpha\in(0,1/2)$, hence it is natural to consider the random function defined in (1.1) as an element of $\mathcal{H}_\alpha[0,1]$ and try to establish its weak convergence to a standard Brownian motion in this function space.

Before stating the results in this direction, let us define for $\alpha\in(0,1)$ the Hölder space $\mathcal{H}_\alpha[0,1]$ of functions $x\colon[0,1]\to\mathbb{R}$ such that $\sup_{s\neq t}|x(s)-x(t)|/|s-t|^\alpha$ is finite. The analogue of the continuity modulus in $C[0,1]$ is w_α, defined by

$$w_\alpha(x,\delta)=\sup_{0<|t-s|<\delta}\frac{|x(t)-x(s)|}{|t-s|^\alpha}. \tag{1.2}$$

We then define $\mathcal{H}_\alpha^0[0,1]$ by $\mathcal{H}_\alpha^0[0,1]:=\{x\in\mathcal{H}_\alpha[0,1],\lim_{\delta\to 0}w_\alpha(x,\delta)=0\}$. We shall essentially work with the space $\mathcal{H}_\alpha^0[0,1]$ which, endowed with $\|x\|_\alpha:=w_\alpha(x,1)+|x(0)|$, is a separable Banach space (while $\mathcal{H}_\alpha[0,1]$ is not separable). Since the canonical embedding $\iota\colon\mathcal{H}_\alpha^o[0,1]\to\mathcal{H}_\alpha[0,1]$ is continuous, each convergence in distribution in $\mathcal{H}_\alpha^o[0,1]$ also takes place in $\mathcal{H}_\alpha[0,1]$.

Let us denote by D_j the set of dyadic numbers in $[0,1]$ of level j, that is,

$$D_0:=\{0,1\},\qquad D_j:=\left\{(2l-1)2^{-j};\ 1\leq l\leq 2^{j-1}\right\},\quad j\geq 1. \tag{1.3}$$

If $r\in D_j$ for some $j\geq 0$, we define $r^+:=r+2^{-j}$ and $r^-:=r-2^{-j}$. For $r\in D_j,\ j\geq 1$, let Λ_r be the function whose graph is the polygonal path joining the points $(0,0)$, $(r^-,0)$, $(r,1)$, $(r^+,0)$ and $(1,0)$. We can decompose each $x\in C[0,1]$ as

$$x=\sum_{r\in D}\lambda_r(x)\Lambda_r=\sum_{j=0}^{+\infty}\sum_{r\in D_j}\lambda_r(x)\Lambda_r, \tag{1.4}$$

and the convergence is uniform on $[0,1]$. The coefficients $\lambda_r(x)$ are given by

$$\lambda_r(x)=x(r)-\frac{x(r^+)+x(r^-)}{2},\quad r\in D_j,\ j\geq 1, \tag{1.5}$$

and $\lambda_0(x)=x(0)$, $\lambda_1(x)=x(1)$.