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Derivative formulae for SDEs driven by multiplicative α-stable-like processes

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Abstract

By using Bismut's approach to the Malliavin calculus with jumps, we establish a derivative formula of Bismut–Elworthy–Li's type for SDEs driven by multiplicative Lévy noises, whose Lévy measure satisfies some order conditions. In particular, α-stable-like noises are allowed. Moreover, we also obtain the sharp gradient estimate in short time for the corresponding transition semigroup provided $\alpha \in (1, 2)$.

Keywords: Derivative formula; Gradient estimate; Stable-like process; Malliavin calculus

1. Introduction

Consider the following stochastic differential equation (abbreviated as SDE) in $\mathbb{R}^d$:

$$X_t = x + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma_1(X_s)\mathrm{d}W_s + \int_0^t \sigma_2(X_{s-})\mathrm{d}L_s, \tag{1.1}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma_1, \sigma_2 : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ satisfy the following assumptions:

$$b, \sigma_1 \text{ and } \sigma_2 \text{ are } C^2\text{-smooth and have bounded first order derivatives,} \tag{1.2}$$

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W_t is a d-dimensional standard Brownian motion, and L_t is a d-dimensional purely jump symmetric Lévy process with Lévy measure $\nu(\mathrm{d}z)$. Let $\Gamma_0 := \{z \in \mathbb{R}^d : 0 < |z| < 1\}$. We assume that $\frac{\nu(\mathrm{d}z)}{\mathrm{d}z}\Big|_{\Gamma_0} = \kappa(z) \in C^1(\Gamma_0; (0, \infty))$ satisfies the following order and bounded conditions: for some $\alpha \in (0, 2)$ and $c_1, c_2 > 0$,

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-2} \int_{|z| \leqslant \varepsilon} |z|^2 \kappa(z) \mathrm{d}z = c_1, \qquad |\nabla \log \kappa(z)| \leqslant c_2 |z|^{-1}, \quad z \in \Gamma_0. \tag{1.3}$$

Notice that the generator of SDE (1.1) is given by

$$\begin{aligned} \mathscr{A} f(x) := &\frac{1}{2} \sum_{ijk} (\sigma_1^{ik} \sigma_1^{jk})(x) \partial_{ij}^2 f(x) + b(x) \cdot \nabla f(x) \\ &+ \mathrm{p.v.} \int_{\mathbb{R}^d - \{0\}} (f(x + \sigma_2(x) z) - f(x)) \nu(\mathrm{d}z), \end{aligned}$$

where ∂_{ij}^2 denotes the second order partial derivative, $\nabla := (\partial_1, \ldots, \partial_d)$ and p.v. stands for Cauchy's principal value. If $\frac{\nu(\mathrm{d}z)}{\mathrm{d}z} = \kappa(z) = a(z)|z|^{-d-\alpha}$ with

$$a(z) = a(-z), \quad 0 < a_0 \leqslant a(z) \leqslant a_1, \qquad |\nabla a(z)| \leqslant a_2,$$

then (1.3) holds. The Lévy process corresponding to this ν will be called α-stable-like process as in [4].

It is well known that under (1.2), SDE (1.1) has a unique solution denoted by $X_t = X_t(x)$, and $\{X_t(x),\ t \geqslant 0\}_{x \in \mathbb{R}^d}$ forms a family of Markov processes and also a C^1-stochastic flow (cf. [7]). Let $\mathcal{B}_b(\mathbb{R}^d)$ be the set of all bounded measurable functions on $\mathbb{R}^d$. Define

$$P_t f(x) := \mathbb{E} f(X_t(x)), \quad f \in \mathcal{B}_b(\mathbb{R}^d). \tag{1.4}$$

Then $(P_t)_{t \geqslant 0}$ is called the Markov semigroup associated with $X_t(x)$. The aim of this work is to establish a formula under (1.2) and (1.3) like

$$\nabla P_t f(x) = \mathbb{E}(f(X_t(x)) H_t(x)), \quad \forall f \in C_b^1(\mathbb{R}^d),$$

where $H_t(x)$ is a vector valued process and ∇ is the weak or distributional derivatives. This type of formula is usually called Bismut–Elworthy–Li's formula in the literature. When $\sigma_2(x) = 0$ and $\sigma_1(x)$ is uniformly nondegenerate, i.e.

$$\|\sigma_1^{-1}\|_\infty := \sup_{x \in \mathbb{R}^d} |\sigma_1^{-1}(x)| < +\infty,$$

where $\sigma_1^{-1}(x)$ denotes the inverse matrix of $\sigma_1(x)$, Bismut [3] and Elworthy–Li [6] proved the following elegant derivative formula for semigroup $P_t f(x)$ by using different approaches:

$$\nabla P_t f(x) = \frac{1}{t} \mathbb{E} \left(f(X_t(x)) \int_0^t (\sigma_1^{-1}(X_s(x)) \nabla X_s(x))^* \mathrm{d}W_s \right),$$

where $(\nabla X_s(x))_{ij} := \partial_j X_s^i(x)$ is the Jacobian matrix, and $*$ denotes the transpose of a matrix. When $\sigma_1(x) = 0$ and $\sigma_2(x)$ is uniformly nondegenerate, and $L_t = W_{S_t}$ is a subordinated Brownian motion, a similar formula was recently proved in [16] (see also [17,12] for the case of additive noises). Moreover, when S_t is an α-stable subordinator, the sharp gradient estimate in short time is derived in [16]. Notice that the proof in [16] is based upon the Malliavin calculus for Brownian