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Dualities in population genetics: A fresh look with new dualities

Gioia Carinci [a], Cristian Giardinà [a,*], Claudio Giberti [b], Frank Redig [c]

[a] *Department of Mathematics, University of Modena and Reggio Emilia, via G. Campi 213/b, 41125 Modena, Italy*
[b] *Department of Sciences and Methods for Engineering, University of Modena and Reggio Emilia, via Giovanni Amendola 2, 42122 Reggio Emilia, Italy*
[c] *Delft Institute of Applied Mathematics, Technische Universiteit Delft, Mekelweg 4, 2628 CD Delft, The Netherlands*

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Abstract

We apply our general method of duality, introduced in Giardinà et al. (2007), to models of population dynamics. The classical dualities between forward and ancestral processes can be viewed as a change of representation in the classical creation and annihilation operators, both for diffusions dual to coalescents of Kingman's type, as well as for models with finite population size.

Next, using $SU(1, 1)$ raising and lowering operators, we find new dualities between the Wright–Fisher diffusion with d types and the Moran model, both in presence and absence of mutations. These new dualities relates two forward evolutions. From our general scheme we also identify self-duality of the Moran model.

Keywords: Mathematical population genetics; Duality; Wright–Fisher model; Moran model; Heisenberg algebra; $SU(1, 1)$ algebra

1. Introduction

Duality is one of the most important techniques in interacting particle systems [25], models of population dynamics [7,26,21], mutually catalytic branching [27] and general Markov process theory [14]. See [23] for a recent review paper containing an extensive list of references, going even back to the work of Lévy.

In all the interacting particle systems models (e.g. symmetric exclusion, voter model, contact process, etc.) about which we know fine details, such as complete ergodic theorems or explicit

* Corresponding author.
E-mail addresses: cristian.giardina@unimore.it, cgiardina@gmail.com (C. Giardinà).

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formulas for time-dependent correlation functions, a non-trivial duality or self-duality relation plays a crucial role. This fact also becomes more and more apparent in recent exact formulas for the transition probabilities of the asymmetric exclusion process [34,3], where often the starting point is a duality of the type first revealed in this context by Schütz in [29]. In [31] the notion of "stochastic integrability" is coined, and related to duality. It is therefore important to gain deeper understanding of "what is behind dualities", i.e., why some processes admit nice dual processes and others not, and where the duality functions come from. This effort is not so much a quest of creating a general abstract framework of duality. It is rather a quest to create a workable constructive approach towards duality, and using this creating both new dualities in known contexts as well as new Markov processes with nice duality properties.

In the works [15,16] duality between two stochastic processes, in the context of interacting particle systems and non-equilibrium statistical mechanics, has been related to a change of representation of an underlying Lie algebra. More precisely, if the generator of a Markov process is built from lowering and raising operators (in physics language creation and annihilation operators) associated to a Lie algebra, then different representations of these operators give rise to processes related to each other by duality. The intertwiner between the different representations is exactly the duality function. Furthermore, self-dualities [17] can be found using symmetries related to the underlying Lie algebra (see also [30,32]).

The fact that generators of Markov processes can be built from raising and lowering operators is a quite natural assumption. In interacting particle systems, the dynamics consists of removing particles at certain places and putting them at other places. If the rates of these transitions are appropriately chosen, then the operators of which the effect is to remove or to add a particle (with appropriate coefficients), together with their commutators, generate a Lie algebra. For diffusion processes the generator is built from a combination of multiplication operators and (partial) derivatives. For specific choices, these correspond to differential operator representations of a Lie algebra. If this Lie algebra also possesses a discrete representation, then this can lead to a duality between a diffusion process and a process of jump type, such as the well-known duality between the Wright–Fisher diffusion and the Kingman's coalescent.

It is the aim of this paper to show that this scheme of finding dual processes via a change of representation can be applied in the context of mathematical population genetics. First, we give a fresh look at the classical dualities between processes of Wright–Fisher type and their dual coalescents. These dualities correspond to a change of representation in the creation and annihilation operators generating the Heisenberg algebra. For population models in the diffusion limit (infinite population size limit) the duality comes from the standard representation of the Heisenberg algebra in terms of the multiplicative and derivative operators $(x, d/dx)$, and another discrete representation, known as the Doi–Peliti representation. The intertwiner is in this case simply the function $D(x, n) = x^n$. In the case of finite population size, dualities arise from going from a finite-dimensional representation (finite dimensional creation and annihilation operators satisfying the canonical commutation relations) to the Doi–Peliti representation. The intertwiner is exactly the hypergeometric polynomial found e.g. in [21,20], and gives duality between the Moran model with finite population size and the Kingman's coalescent.

Next we use the $SU(1, 1)$ algebra to find previously unrevealed dualities between the discrete Moran model and the Wright–Fisher diffusion, as well as self-duality of the discrete Moran model.

These are in fact applications of the previously found dualities between the Brownian energy process and the symmetric inclusion process, as well as the self-duality of the symmetric inclusion process, which we have studied in another context in [15–17]. Put into the context