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A topology for limits of Markov chains

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Abstract

In the investigation of limits of Markov chains, the presence of states which become instantaneous states in the limit may prevent the convergence of the chain in the Skorohod topology. We present in this article a weaker topology adapted to handle this situation. We use this topology to derive the limit of random walks among random traps and sticky zero-range processes.

Keywords: Metastability; Skorohod topology

1. Introduction

Some Markov chains can be approximated by Markovian dynamics evolving in a contracted state space. This is the case of certain zero-range models, whose dynamics can be approximated by the one of a random walk, [3,13], and of some polymer models whose evolution can be approximated by a two-state Markov chain [6,7,5,12]. We proposed in [1,4] a formal definition of this phenomenon. In analogy to statistical mechanic models, we named these processes metastable Markov chains, and we developed tools to prove the convergence (of the projection on a contracted state space) of these chains to Markovian dynamics. The erratic behavior of the projected chain in very short time intervals precludes convergence in the Skorohod topology. We introduce in this article a topology in which the convergence takes place. This topology might be useful in other contexts. For example, in the investigation of limits of Markov chains when some

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states become instantaneous states in the limit, that is, states whose jump rates become infinite. To explain the topological problem created by the asymptotic instantaneous states and to present the main results of the article, we examine in this introduction sticky zero-range processes and random walks among random traps.

1.1. Zero-range processes

Fix a finite set $S_L = \{1, \dots, L\}$, and denote by $E_{L,N}$, $N \ge 1$, the configurations obtained by distributing N particles on S_L:

$$E_{L,N} := \Big\{\eta \in \mathbb{N}^{S_L} : \sum_{x\in S_L} \eta_x = N\Big\}.$$

Consider an irreducible, continuous-time random walk $\{Z(t) : t \ge 0\}$ on S_L which jumps from x to y at a rate $r(x, y)$ which is reversible with respect to the uniform measure, $r(x, y) = r(y, x)$, $x, y \in S_L$. Fix $\alpha > 1$, and let $g : \mathbb{N} \to \mathbb{R}$ be given by

$$g(0) = 0, \qquad g(1) = 1, \quad \text{and} \quad g(n) = \Big(\frac{n}{n-1}\Big)^{\alpha}, \quad n \ge 2,$$

so that $\prod_{i=1}^{n} g(i) = n^{\alpha}$, $n \ge 1$.

Denote by $\{\eta^N(t) : t \ge 0\}$ the zero-range process on S_L in which a particle jumps from a site x, occupied by k particles, to a site y at rate $g(k)r(x, y)$. The generator of this Markov chain $\eta(t) = \eta^N(t)$, represented by L_N, acts on functions $F : E_{L,N} \to \mathbb{R}$ as

$$(L_N F)(\eta) = \sum_{\substack{x,y\in S_L\\ x\neq y}} g(\eta_x)\, r(x, y) \big\{F(\sigma^{x,y}\eta) - F(\eta)\big\},$$

where $\sigma^{x,y}\eta$ is the configuration obtained from η by moving a particle from x to y:

$$(\sigma^{x,y}\eta)_z = \begin{cases} \eta_x - 1 & \text{for } z = x \\ \eta_y + 1 & \text{for } z = y \\ \eta_z & \text{otherwise.} \end{cases}$$

Fix a sequence $\{\ell_N : N \ge 1\}$ such that $1 \ll \ell_N \ll N$, where $a_N \ll b_N$ means that $a_N/b_N \to 0$. For x in S_L, let

$$\mathcal{E}^x_N := \big\{\eta \in E_{L,N} : \eta_x \ge N - \ell_N\big\}.$$

Since $\ell_N/N \to 0$, on each set $\mathcal{E}^x_N$ the proportion of particles at $x \in S_L$, η_x/N, is almost one. Assume that N is large enough so that $\mathcal{E}^x_N \cap \mathcal{E}^y_N = \varnothing$ for $x \neq y$, and consider the partition

$$E_{L,N} = \mathcal{E}^1_N \cup \cdots \cup \mathcal{E}^L_N \cup \Delta_N,$$

where Δ_N is the set of configurations which do not belong to the set $\mathcal{E}_N = \cup_{1\le x\le N} \mathcal{E}^x_N$.

Denote by π_N the stationary measure of the zero-range dynamics $\eta(t)$. We proved in [3] that the measure π_N is concentrated on the set $\mathcal{E}_N$:

$$\lim_{N\to\infty} \pi_N(\mathcal{E}^x_N) = \frac{1}{L}.$$

For $N > L$, let the projection $\Psi_N : E_{L,N} \to \{1, \dots, L\} \cup \{N\}$ be defined by

$$\Psi_N(\eta) = \sum_{x=1}^{L} x\, \mathbf{1}\{\eta \in \mathcal{E}^x_N\} + N\mathbf{1}\{\eta \in \Delta_N\},$$