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# Level set percolation for random interlacements and the Gaussian free field

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**Abstract**

We consider continuous-time random interlacements on $\mathbb{Z}^d$, $d \geq 3$, and investigate the percolation model where a site $x$ of $\mathbb{Z}^d$ is occupied if the total amount of time spent at $x$ by all the trajectories of the interlacement at level $u \geq 0$ exceeds some constant $\alpha \geq 0$, and empty otherwise. We also examine percolation properties of empty sites. A recent isomorphism theorem (Sznitman, 2012) enables us to "translate" some of the relevant questions into the language of level-set percolation for the Gaussian free field on $\mathbb{Z}^d$, $d \geq 3$, about which new insights of independent interest are also gained.


*Keywords:* Percolation; Long-range dependence; Gaussian free field; Random interlacements; Level sets

## 0. Introduction

In the present work, we consider the field of occupation times for continuous-time random interlacement at level $u \geq 0$ on $\mathbb{Z}^d$, $d \geq 3$, and investigate the percolative properties of the random subset of $\mathbb{Z}^d$ obtained by keeping only those sites at which the occupation time exceeds some given cut-off value $\alpha \geq 0$. We also consider the percolative properties of the complement of this set in $\mathbb{Z}^d$. Our main interest is to infer for which values of the parameters $(u, \alpha)$ these random sets percolate. A recent isomorphism theorem [15] relates the field of occupation times for continuous-time random interlacements on $\mathbb{Z}^d$, $d \geq 3$ (and more generally, on any transient weighted graph) to the Gaussian free field on the same graph. We will exploit this correspondence as a transfer mechanism to reformulate some of the problems in terms of

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questions regarding level-set percolation for the Gaussian free field. This will allow us to use certain renormalization techniques recently developed in this context in [11]. Additionally, we derive new results concerning "two-sided" level-set percolation for the Gaussian free field on $\mathbb{Z}^d$, $d \geq 3$, where, in contrast to (0.2) of [11] (see also [2]), the level sets consist of those sites at which the *absolute value* of the corresponding field variable exceeds a certain level $h \geq 0$.

We now describe our results and refer to Section 1 for details. We consider continuous-time random interlacements on $\mathbb{Z}^d$, $d \geq 3$. Somewhat informally, this model can be defined as a cloud of simple random walk trajectories modulo time-shift on $\mathbb{Z}^d$ constituting a Poisson point process, where a non-negative parameter $u$ appearing multiplicatively in the intensity measure regulates how many paths enter the picture (we defer a precise definition to the next section, see the discussion around (1.8)). For any $u \geq 0$ and $\alpha \geq 0$, we introduce the (random) subsets of $\mathbb{Z}^d$

$$\mathcal{I}^{u,\alpha} = \{x \in \mathbb{Z}^d; L_{x,u} > \alpha\}, \qquad \mathcal{V}^{u,\alpha} = \{x \in \mathbb{Z}^d ; L_{x,u} \leq \alpha\} = \mathbb{Z}^d \setminus \mathcal{I}^{u,\alpha}, \tag{0.1}$$

where $(L_{x,u})_{x\in\mathbb{Z}^d}$ denotes the field of occupation times at level $u$, see (1.15), and ask for which values of the parameters $u$ and $\alpha$ these sets percolate. Note that for all $u \geq 0$, $\mathcal{I}^{u,0}$ corresponds to the (discrete-time) interlacement set at level $u$ introduced in (0.7) of [14] (see also (1.9) and (1.16) below) and $\mathcal{V}^{u,0}$ to the according vacant set. Before addressing the core issue of describing the phase diagrams for percolation of the random sets $\mathcal{I}^{u,\alpha}$ and $\mathcal{V}^{u,\alpha}$, as $u$ and $\alpha$ vary, we prove uniqueness of the infinite clusters, whenever they exist. More precisely, we show in Corollary 2.6 that for all $u \geq 0$, $\alpha > 0$ and $d \geq 3$,

$$\mathbb{P}\text{-a.s., } \mathcal{I}^{u,\alpha} \text{ and } \mathcal{V}^{u,\alpha} \text{ contain } \textit{at most one} \text{ infinite connected component,} \tag{0.2}$$

where $\mathbb{P}$ denotes the law of the interlacement point process, as defined below (1.8). For $\alpha = 0$, (0.2) is already known and follows from [14], Corollary 2.3, and [17], Theorem 1.1.

Our main results concern the existence/absence of infinite clusters inside $\mathcal{I}^{u,\alpha}$ and $\mathcal{V}^{u,\alpha}$, in terms of the parameters $u$ and $\alpha$. Let us define the functions

$$\eta^{\mathcal{I}}(u,\alpha) = \mathbb{P}\big[0 \overset{\mathcal{I}^{u,\alpha}}{\longleftrightarrow} \infty\big], \qquad \eta^{\mathcal{V}}(u,\alpha) = \mathbb{P}\big[0 \overset{\mathcal{V}^{u,\alpha}}{\longleftrightarrow} \infty\big], \quad \text{for } u \geq 0, \alpha \geq 0, \tag{0.3}$$

to denote the probabilities that 0 lies in an infinite cluster of $\mathcal{I}^{u,\alpha}$ and $\mathcal{V}^{u,\alpha}$, respectively. Observing that $\eta^{\mathcal{I}}(u,\alpha)$ is decreasing in $\alpha$ for every (fixed) value of $u \geq 0$, it is sensible to introduce the critical parameter

$$\alpha_*(u) = \inf\{\alpha \geq 0; \eta^{\mathcal{I}}(u,\alpha) = 0\} \in [0,\infty], \quad \text{for } u \geq 0 \tag{0.4}$$

(with the convention $\inf \emptyset = \infty$). It is not difficult to see that the function $\alpha_*(\cdot)$ is non-decreasing, see (5.1) below. Our main results regarding percolation of the sets $\mathcal{I}^{u,\alpha}$ state that

$$0 < \alpha_*(u) < \infty, \quad \text{for all } u > 0 \text{ and } d \geq 3 \tag{0.5}$$

(see Theorem 3.1 for positivity of $\alpha_*(u)$ and Theorem 5.1 for finiteness). In words, the sets $\mathcal{I}^{u,\alpha}$ exhibit a non-trivial percolation phase transition as $\alpha$ varies, for every (fixed) positive value of $u$.

In a similar vein, for $\mathcal{V}^{u,\alpha}$, we introduce the critical parameter

$$u_*(\alpha) = \inf\{u \geq 0; \eta^{\mathcal{V}}(u,\alpha) = 0\} \in [0,\infty], \quad \text{for } \alpha \geq 0, \tag{0.6}$$

which is well-defined since $\eta^{\mathcal{V}}(\cdot,\alpha)$ is non-increasing for every value of $\alpha \geq 0$ (we will comment on the asymmetry in the role of $u$ and $\alpha$ in (0.4) and (0.6) below; see the discussion following (0.14)). It is an easy matter to verify that the function $u_*(\cdot)$ is non-decreasing, see (5.6), and that