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Random walks in cones: The case of nonzero drift

Jetlir Duraj *

Mathematical Institute, University of Munich, Theresienstrasse 39, D-80333 Munich, Germany

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Abstract

We consider multidimensional discrete valued random walks with nonzero drift killed when leaving general cones of the euclidean space. We find the asymptotics for the exit time from the cone and study weak convergence of the process conditioned on not leaving the cone. We get quasistationarity of its limiting distribution. Finally we construct a version of the random walk conditioned to never leave the cone.

Keywords: Random walk; Exit time; Cones; Conditioned process; Quasistationary distribution

1. Introduction, results and discussion

1.1. Motivation

We want to study the multidimensional counterpart of the following one dimensional problem.

Let $S(n)$ be a real valued random walk with negative drift started at some $x \in (0, +\infty)$. Find the asymptotics of the exit time $\tau_x = \inf\{n \geq 0 : S(n) \leq 0\}$. This is by now a classical result. For example, in [5] the asymptotics is found if the jump of the random walk fulfills the following Cramér-type condition: $R(h) = \mathbb{E}[e^{hX}]$ is finite in some $[0, B]$ for $B \leq \infty$, and $0 < \lim_{h\uparrow B} \frac{R'(h)}{R(h)} \leq \infty$. The asymptotics is then

$$\mathbb{P}(\tau_x > n) \sim V(x)\mu^{-n} n^{-\frac{3}{2}} \quad \text{as } n \to \infty. \tag{1}$$

* Tel.: +49 89 2180 4170; fax: +49 89 2180 4043.
E-mail address: jetlir.duraj@mathematik.uni-muenchen.de.

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Here, $\mu = \frac{1}{\mathbb{E}[e^{h_0 X}]}$ and h_0 is the unique solution of $R'(h) = 0$ and the assumption on the jump implies in particular that it has a finite second moment. For two real-valued sequences, by the notation $a(n) \sim b(n)$, as $n \to \infty$, we mean the property $\lim_{n\to\infty} \frac{a(n)}{b(n)} = 1$.

Note that the positive open real halfline is a cone. Our main aim in this paper is to find an analogue to (1) for random walks in dimensions greater than one, killed when leaving cones in the respective euclidean space and to derive some weak convergence results for the conditioned process.

Even though the idea of a Cramér condition and use of an exponential change of measure will be helpful for the multidimensional case as well, methodologically the multidimensional problem is different from the one dimensional case. Therefore we first briefly recall the main idea of the study of the latter problem, as it is done in [5].

Its study is facilitated by the obvious relation $\mathbb{P}(\tau_x > n) = \mathbb{P}(L_n \geq -x)$ with $L_n = \min_{i=1..n} S(i)$. This makes possible the use of the following classical relations (see also the similar equations (2.5) and (2.6) in [5])

$$\sum_{n\geq 0} s^n \mathbb{E}[e^{sL_n}] = \exp\left\{\sum_{n\geq 1} s^n a_n(s)\right\} \tag{2}$$

and

$$\sum_{n\geq 0} s^n \mathbb{P}(\tau_0 > n) = \exp\left\{\sum_{n\geq 1} s^n a_n\right\}, \tag{3}$$

where $a_n(t) = \frac{1}{n}\{\mathbb{E}[e^{tS(n)}, S(n) < 0] + \mathbb{P}(S_n \geq 0)\}$ and $a_n = \frac{1}{n}\mathbb{P}(S_n \geq 0)$. Then getting the asymptotics of $a_n(t)$ and a_n yields the asymptotics of the exit time. To this aim one makes use of a change of measure, for example for a_n we have

$$\begin{aligned} a_n &= \frac{1}{n}\int_0^\infty \mathbb{P}(S_n \in dx) \\ &= \frac{1}{n}\int_0^\infty e^{-hx}(\mathbb{E}[e^{hx}])^n \mathbb{P}(\hat{S}_n \in dx) \\ &= \frac{(\mathbb{E}[e^{hx}])^n}{n}\mathbb{E}[e^{-\hat{S}_n}, \hat{S}_n \geq 0]. \end{aligned}$$

Here $\hat{S}_n$ is the driftless random walk gained from an exponential change of measure of X through the density $\frac{e^{hx}}{\mathbb{E}[e^{hx}]}$. It is then easy to see, for example if the random walk is discrete by expanding and using a local limit theorem, that the expectation in the last line is asymptotically $\frac{const}{\sqrt{n}}$.

As already mentioned, there is no hope of some similarly helpful relation as (2) and (3) for the multidimensional case. A way of attacking the multidimensional case is supplied by the recent work [4]. In it the authors study the asymptotics of the exit time from a general multidimensional cone for the case of driftless random walks. They use then the asymptotics of the exit time and several sharp probabilistic inequalities to establish local limit theorems for lattice valued driftless random walks, killed when leaving the cone. One could use some of their results after reducing the case of a nonzero drift to that of the zero drift. For this we impose a Cramér condition and use an exponential change of measure to turn the nonzero drift random walk into a driftless one already at the beginning. Nevertheless, it turns out that one has to refine and specialize some