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Comparison inequalities on Wiener space

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Abstract

We define a covariance-type operator on Wiener space: for F and G two random variables in the Gross–Sobolev space $\mathbb{D}^{1,2}$ of random variables with a square-integrable Malliavin derivative, we let $\Gamma_{F,G} := \left\langle DF, -DL^{-1} G\right\rangle$, where D is the Malliavin derivative operator and L^{-1} is the pseudo-inverse of the generator of the Ornstein–Uhlenbeck semigroup. We use Γ to extend the notion of covariance and canonical metric for vectors and random fields on Wiener space, and prove corresponding non-Gaussian comparison inequalities on Wiener space, which extend the Sudakov–Fernique result on comparison of expected suprema of Gaussian fields, and the Slepian inequality for functionals of Gaussian vectors. These results are proved using a so-called smart-path method on Wiener space, and are illustrated via various examples. We also illustrate the use of the same method by proving a Sherrington–Kirkpatrick universality result for spin systems in correlated and non-stationary non-Gaussian random media.

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1. Introduction

The canonical metric of a centered field G on an index set T is the square root of the quantity $\delta_G^2(s,t) = \mathbf{E}\left[(G_t - G_s)^2\right]$, $s,t \in T$. When G is Gaussian, this δ^2 characterizes

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much of G's distribution, and is useful in various contexts for estimating G's behavior, from its modulus of continuity, to its expected supremum; see [1] for an introduction. The canonical metric, together with the variances of G, are of course equivalent to the covariance function $Q_G(s,t) = \mathbf{E}[G_t G_s]$, which defines G's law when G is Gaussian. In this article, we concentrate on comparison results for expectations of suprema and other types of functionals, beyond the Gaussian context, by using an extension of the concepts of covariance and canonical metric on Wiener space. We introduce these concepts now. For the details of analysis on Wiener space needed for the next definitions, including the spaces $\mathbb{D}^{1,p}$ ($p \geqslant 1$) and the operators D and L^{-1}, see Chapter 1 in [14] or Chapter 2 in [10]. The notion of a 'separable random field' is formally defined e.g. in [2, p. 8].

Definition 1.1. Consider an isonormal Gaussian process W defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and associated with the real separable Hilbert space $\mathfrak{H}$: recall that this means that $W = \{W(h) : h \in \mathfrak{H}\}$ is a centered Gaussian family such that $\mathbf{E}[W(h)W(k)] = \langle h, k\rangle_{\mathfrak{H}}$. Let $\mathbb{D}^{1,2}$ be the Gross–Sobolev space of random variables F with a square-integrable Malliavin derivative, i.e. such that $DF \in L^2(\Omega \times \mathfrak{H})$. We denote the generator of the associated Ornstein–Uhlenbeck operator by L. For a pair of random variables $F, G \in \mathbb{D}^{1,2}$, we define a covariance-type operator by

$$\Gamma_{F,G} := \langle DF, -DL^{-1}G\rangle_{\mathfrak{H}}. \tag{1.1}$$

Let $F = \{F_t\}_{t\in T}$ be a separable random field on an index set T, such that $F_t \in \mathbb{D}^{1,2}$ for each $t \in T$. The analogue for the operator Γ of the covariance of F is denoted by

$$\Gamma_F(s,t) := \Gamma_{F_s,F_t} = \langle D(F_t), -DL^{-1}(F_s)\rangle_{\mathfrak{H}}. \tag{1.2}$$

The analogue for Γ of the canonical metric δ^2 of F is denoted by

$$\Delta_F(s,t) := \langle D(F_t - F_s), -DL^{-1}(F_t - F_s)\rangle_{\mathfrak{H}}. \tag{1.3}$$

Remark 1.2. (i) When $F = \{F_t\}_{t\in T}$ is in the first Wiener chaos, and hence is a centered Gaussian field, Γ_F coincides with its covariance function Q_F.

(ii) In general, the random variable $\Delta_F(s,t)$ is not positive. However, according e.g. to [9, Proposition 3.9], one has that $\mathbf{E}[\Delta_F(s,t)|F_t - F_s] \geqslant 0$, a.s.-$\mathbf{P}$.

(iii) In general, we do not have $\Gamma_{F,G} = \Gamma_{G,F}$. However, Γ does extend the notion of covariance for centered random variables, in the sense that $E[\Gamma_{F,G}] = E[FG]$. More generally, if F and G are in the same Wiener chaos, then $\Gamma_{F,G} = \Gamma_{G,F}$, but this symmetry does not extend in general beyond such special cases.

The extension of the concept of covariance function given above in (1.1) appeared in [3,11], respectively in the study of densities of random vectors and of multivariate normal approximations, both on Wiener space. Comparison results on Wiener space have, in the past, focused on concentration or Poincaré inequalities: see [19]. Recently, the scalar analogue of the covariance operator above, i.e. $\Gamma_{F,F}$, was exploited to derive sharp tail comparisons on Wiener space, in [13,20].

The two main types of comparison results we will investigate herein are those of Sudakov–Fernique type and those of Slepian type. See [1,2] for details of the classical proofs.

In the basic *Sudakov–Fernique* inequality, one considers two centered separable Gaussian fields F and G on T, such that $\delta_F^2(s,t) \geqslant \delta_G^2(s,t)$ for all $s,t \in T$; then $\mathbf{E}\left[\sup_T F\right] \geqslant \mathbf{E}\left[\sup_T G\right]$. Here T can be any index set, as long as the laws of F and G can be determined