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A general correlation inequality and the Almost Sure Local Limit Theorem for random sequences in the domain of attraction of a stable law[☆]

Rita Giuliano[a,*], Zbigniew S. Szewczak[b]

[a] *Dipartimento di Matematica "L. Tonelli", Università di Pisa, Largo Bruno Pontecorvo 5, I-56127 Pisa, Italy*
[b] *Nicolaus Copernicus University, Faculty of Mathematics and Computer Science, ul. Chopina* 12/18 *87-100 Toruń, Poland*

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Abstract

In the present paper we obtain a new correlation inequality and use it for the purpose of extending the theory of the Almost Sure Local Limit Theorem to the case of lattice random sequences in the domain of attraction of a stable law. In particular, we prove ASLLT in the case of the normal domain of attraction of α-stable law, $\alpha \in (1, 2)$.

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1. Introduction

In the recent paper [10], the author proves a correlation inequality and an Almost Sure Local Limit Theorem (ASLLT) for i.i.d. square integrable random variables taking values in a lattice.

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* Corresponding author.
E-mail addresses: giuliano@dm.unipi.it (R. Giuliano), zssz@mat.uni.torun.pl (Z.S. Szewczak).

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The sequence of partial sums of such variables are of course in the domain of attraction of the normal law, which is stable of order $\alpha = 2$.

The aim of the present paper is to give an analogous correlation inequality (Theorem 3.1) for the more general case of random sequences in the domain of attraction of a stable law of order $\alpha \leqslant 2$ and to apply it for the purpose of extending the theory of ASLLT. Notice that in our situation the summands need not be square integrable. Our correlation inequality turns out to be of the typical form needed in the theory of Almost Sure (Central and Local) Limit Theorems (see Corollary 3.3 and Remark 3.4). Our work is based on a careful use of the form of the characteristic function, and is completely different from the one used in [10] (McDonald's method of extraction of the Bernoulli part of a random variable).

2. The assumptions and some preliminaries

In this paper we shall be concerned with a sequence of i.i.d. random variables $(X_n)_{n\geq 1}$ such that their common distribution F is in the domain of attraction of G, where G is a stable distribution with exponent α ($0 < \alpha \leqslant 2$, $\alpha \neq 1$). This means that, for a suitable choice of constants a_n and b_n, the distribution of

$$T_n := \frac{X_1 + \cdots X_n - a_n}{b_n}$$

converges weakly to G. It is well known (see [6], p. 46) that in such a case we have $b_n = L(n)n^{1/\alpha}$, where L is slowly varying in Karamata's sense. For $\alpha > 1$ we shall assume that X_1 is centered; by Remark 2, p. 402 of [1], this implies that $a_n = 0$, for every α.

We shall suppose that X_1 takes values in the lattice $\mathcal{L}(a, d) = \{a + kd,\ k \in \mathbb{Z}\}$ where d is the maximal span of the distribution; hence $S_n := X_1 + \cdots X_n$ takes values in the lattice $\mathcal{L}(na, d) = \{na + kd,\ k \in \mathbb{Z}\}$.

For every n, let κ_n be a number of the form $na + kd$ and let

$$\lim_{n\to\infty} \frac{\kappa_n}{b_n} = \kappa.$$

Observe that Theorem 4.2.1, p. 121 in [6], implies that

$$\sup_n \Big\{ \sup_k b_n P(S_n = k) \Big\} = C < \infty. \tag{1}$$

Throughout this paper we assume that

$$\begin{aligned} &x^\alpha P(X > x) = \big(c_1 + o(1)\big) l(x); \\ &x^\alpha P(X \leqslant -x) = \big(c_2 + o(1)\big) l(x), \quad \alpha \in (0, 2], \end{aligned} \tag{2}$$

where l is slowly varying as $x \to \infty$ and c_1 and c_2 are two suitable non-negative constants, $c_1 + c_2 > 0$, related to the stable distribution G.

Let ϕ be the characteristic function of F. By [1], Theorem 1, for $\alpha \neq 1$ it has the form

$$\phi(t) = \exp\left\{ -c|t|^\alpha h\Big(\frac{1}{|t|}\Big)\Big(1 - i\beta \operatorname{sign}(t) \tan\frac{\pi\alpha}{2}\Big) + o\left(|t|^\alpha h\Big(\frac{1}{|t|}\Big)\right) \right\}, \tag{3}$$

where $c = \Gamma(1-\alpha)(c_1 + c_2)\cos\frac{\pi\alpha}{2} > 0$ and $\beta = \frac{c_1 - c_2}{c_1 + c_2} \in [-1, 1]$ are two constants and $h(x) = l(x)$ if $\alpha \in (0, 2)$ and $c = \frac{1}{2}$, $\beta = 0$, $h(x) = E[X^2 1_{\{|X| \leqslant x\}}]$ if $\alpha = 2$. This formula