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Generalized Hermite processes, discrete chaos and limit theorems

Shuyang Bai, Murad S. Taqqu*

Department of Mathematics and Statistics, 111 Cumminton Street, Boston, MA, 02215, United States

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Abstract

We introduce a broad class of self-similar processes $\{Z(t), t \geq 0\}$ called generalized Hermite processes. They have stationary increments, are defined on a Wiener chaos with Hurst index $H \in (1/2, 1)$, and include Hermite processes as a special case. They are defined through a homogeneous kernel g, called the "generalized Hermite kernel", which replaces the product of power functions in the definition of Hermite processes. The generalized Hermite kernels g can also be used to generate long-range dependent stationary sequences forming a discrete chaos process $\{X(n)\}$. In addition, we consider a fractionally-filtered version $Z^{\beta}(t)$ of $Z(t)$, which allows $H \in (0, 1/2)$. Corresponding non-central limit theorems are established. We also give a multivariate limit theorem which mixes central and non-central limit theorems.

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Keywords: Long memory; Discrete chaos; Wiener chaos; Limit theorem

1. Introduction

A stochastic process $\{X(t), t \geq 0\}$ with finite variance taking values in $\mathbb{R}$ is said to be *self-similar* if there is a constant called *Hurst coefficient* $H > 0$, such that for any scaling factor $a > 0$, $X(at) \overset{f.d.d.}{=} a^H X(t)$, where $\overset{f.d.d.}{=}$ means equality in finite-dimensional distributions. If a self-similar process $\{X(t), t \geq 0\}$ has also stationary increments, namely, if for any $h \geq 0$, $\{Y(t) := X(t+h) - X(t), t \geq 0\}$ is a stationary process, then we say that $\{X(t), t \geq 0\}$ is

* Corresponding author. Tel.: +1 6173533022; fax: +1 6173538100.
E-mail addresses: bsy9142@bu.edu (S. Bai), murad@bu.edu, bumastat@gmail.com (M.S. Taqqu).

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H-sssi. The natural range of H is (0, 1), which implies $\mathbb{E}X(t) = 0$ for all $t \geq 0$. We refer the reader to Chapter 3 of Embrechts and Maejima [7] for details.

The fundamental theorem of Lamperti [12] states that H-sssi processes are the only possible limit laws of normalized partial sum of stationary sequences, that is, if

$$\frac{1}{A(N)} \sum_{n=1}^{[Nt]} X(n) \stackrel{f.d.d.}{\longrightarrow} Y(t)$$

and $A(N) \to \infty$ as $N \to \infty$, where $\{X(n)\}$ is stationary, then $\{Y(t), t \geq 0\}$ has to be H-sssi for some $H > 0$, and $A(N)$ has to be regularly varying with exponent H. The notation $\stackrel{f.d.d.}{\longrightarrow}$ stands for convergence in finite-dimensional distributions (f.d.d.).

The best known example of Lamperti's fundamental theorem is when $\{X(n)\}$ is i.i.d. or a *short-range dependent* (SRD) sequence, then the limit $Y(t)$ is Brownian motion which is $\frac{1}{2}$-sssi. If $\{X(n)\}$ has *long-range dependence* (LRD), the limit $Y(t)$ is often H-sssi with $H > 1/2$. The most typical H-sssi process is fractional Brownian motion $B_H(t)$, but there are also non-Gaussian processes, e.g., *Hermite processes* (Taqqu [29] and Dobrushin and Major [6]). The Hermite process of order 1 is fractional Brownian motion, but when the order is greater than or equal to 2, its law belongs to higher-order Wiener chaos (see, e.g., Peccati and Taqqu [23]) and is thus non-Gaussian.

The Hermite processes have attracted a lot of attention. The first-order Hermite process, namely fractional Brownian motion, has been studied intensively by numerous researchers since its popularization by Mandelbrot and Van Ness [19], and we refer the reader to a recent monograph Nourdin [21] and the references therein. The second-order Hermite process, namely the Rosenblatt process, is also investigated in a number of papers. Recent works include Tudor [30], Bardet and Tudor [2], Veillette and Taqqu [32] and Maejima and Tudor [15,16]. Hermite processes frequently appear in statistical inference problems involving LRD, e.g., Lévy-Leduc et al. [13] and Dehling et al. [4].

It is interesting to note that when the stationary sequence $\{X(n)\}$ is LRD, one can obtain in the limit a much richer class of processes, whereas in the SRD case, one obtains only Brownian motion. The type of limit theorems involving H-sssi processes other than Brownian motion are often called *non-central limit theorems*. While Hermite processes are the main examples of H-sssi processes obtained as the limit of partial sum of the finite-variance LRD sequence, there are very few other limit H-sssi processes which have been considered, with some exceptions Rosenblatt [25] and Major [18].

In this paper, we introduce a broad class of H-sssi ($H > 1/2$) processes $\{Z(t), t \geq 0\}$ with their laws in Wiener chaos, which includes the Hermite processes as a special case. These processes are defined as $Z(t) = I_k(h_t)$, where $I_k(\cdot)$ denotes the k-tuple Wiener–Itô integral, and

$$h_t(x_1, \ldots, x_k) := \int_0^t g(s - x_1, \ldots, s - x_k) 1_{\{s>x_1,\ldots,s>x_k\}} ds,$$

with g being some suitable homogeneous function on $\mathbb{R}_+^k$ called the *generalized Hermite kernel*. For example,

$$g(x_1, \ldots, x_k) = \max\left(\frac{x_1 \ldots x_k}{x_1^{k-\alpha} + \cdots + x_k^{k-\alpha}}, x_1^{\alpha/k} \ldots x_k^{\alpha/k} \right),$$
$$\mathbf{x} \in \mathbb{R}_+^k, \alpha \in (-k/2 - 1/2, -k/2). \tag{1}$$