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# Energy of taut strings accompanying Wiener process

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**Abstract**

Let $W$ be a Wiener process. For $r > 0$ and $T > 0$ let $I_W(T, r)^2$ denote the minimal value of the energy $\int_0^T h'(t)^2 dt$ taken among all absolutely continuous functions $h(\cdot)$ defined on $[0, T]$, starting at zero and satisfying

$$W(t) - r \le h(t) \le W(t) + r, \qquad 0 \le t \le T.$$

The function minimizing energy is a taut string, a classical object well known in Variational Calculus, in Mathematical Statistics, and in a broad range of applications. We show that there exists a constant $\mathcal{C} \in (0, \infty)$ such that for any $q > 0$

$$\frac{r}{T^{1/2}} I_W(T, r) \xrightarrow{L_q} \mathcal{C}, \quad \text{as } \frac{r}{T^{1/2}} \to 0,$$

and for any fixed $r > 0$,

$$\frac{r}{T^{1/2}} I_W(T, r) \xrightarrow{\text{a.s.}} \mathcal{C}, \quad \text{as } T \to \infty.$$

Although precise value of $\mathcal{C}$ remains unknown, we give various theoretical bounds for it, as well as rather precise results of computer simulation.

While the taut string clearly depends on entire trajectory of $W$, we also consider an adaptive version of the problem by giving a construction (called Markovian pursuit) of a random function $h(t)$ based only on the values $W(s), s \le t$, and having minimal asymptotic energy. The solution, i.e. an optimal

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pursuit strategy, turns out to be related with a classical minimization problem for Fisher information on the bounded interval.


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## 0. Introduction

Given a time interval $[0, T]$ and two functional boundaries $g_1(t) < g_2(t),\ 0 \le t \le T$, the *taut string* is a function $h_*$ that for any (!) convex function $\varphi$ provides minimum for the functional

$$F_\varphi(h) := \int_0^T \varphi(h'(t))\, dt$$

among all absolutely continuous functions $h$ with given starting and final values and satisfying

$$g_1(t) \le h(t) \le g_2(t), \quad 0 \le t \le T.$$

The list of simultaneously optimized functionals includes energy $\int_0^T h'(t)^2 dt$, variation $\int_0^T |h'(t)| dt$, graph length $\int_0^T \sqrt{1 + h'(t)^2} dt$, etc.

The first instance of taut strings that we have found in the literature is in G. Dantzig's paper [6]. Dantzig notes there that the problem under study and its solution was discussed in R. Bellman's seminar at RAND Corporation in 1952. The taut strings were later used in Statistics, see [18,7]. In the book [21, Chapter 4, Section 4.4], taut strings are considered in connection with problems in image processing. Quite recently, taut strings were applied to a buffer management problem in communication theory, see [22].

In this article, we study the energy of the taut string going through the tube of constant width constructed around sample path of a Wiener process $W$, i.e. for some $r > 0$ we let $g_1(t) := W(t) - r$, $g_2(t) := W(t) + r$, see Fig. 1.

We focus attention on the behavior in a long run: we show that when $T \to \infty$, the taut string spends asymptotically constant amount of energy $\mathcal{C}^2$ per unit of time. Precise assertions are given in Theorems 1.1 and 1.2. The constant $\mathcal{C}$ shows *how much energy an absolutely continuous function must spend if it is bounded to stay within a certain distance from the non-differentiable trajectory of $W$*.

Although precise value of $\mathcal{C}$ remains unknown, we give various theoretical bounds for it in Section 4, as well as the results of computer simulation in Section 6. The latter suggests $\mathcal{C} \approx 0.63$.

If we take the pursuit point of view, considering $h(\cdot)$ as a trajectory of a particle moving with finite speed and trying to stay close to a Brownian particle, then it is much more natural to consider constructions that define $h(t)$ in an adaptive way, i.e. on the base of the known $W(s),\ s \le t$. Recall that the taut string depends on the entire trajectory $W(s),\ s \le T$, hence it does not fit the adaptive setting. In view of Markov property of $W$, the reasonable pursuit strategy for $h(t)$ is to move towards $W(t)$ with the speed depending on the distance $|h(t) - W(t)|$. In this class of algorithms we find an optimal one in Section 5. The corresponding function spends in average $\frac{\pi}{2} \approx 1.57$ units of energy per unit of time. Comparing of two constants shows that we have to pay more than double price for not knowing the future of the trajectory of $W$. To our