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Conformal restriction: The radial case

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Abstract

We describe all random sets that satisfy the radial conformal restriction property, therefore providing the analogue in the radial case of results of Lawler, Schramm and Werner in the chordal case.

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1. Introduction

The present paper is a write-up of the "radial" counterpart of some of the results derived in the "chordal" setting in the paper [6] by Lawler, Schramm and Werner. The goal is to describe the laws of all random sets that satisfy a certain radial conformal restriction property.

Let us describe without further ado this property, and the main result of the present paper: Consider the unit disc $\mathbb{U}$ and we fix a boundary point 1 and an interior point the origin. We will study closed random subsets K of $\overline{\mathbb{U}}$ such that:

- K is connected, $\mathbb{C} \setminus K$ is connected, $K \cap \partial\mathbb{U} = \{1\}$, $0 \in K$.
- For any closed subset A of $\overline{\mathbb{U}}$ such that $A = \overline{\mathbb{U} \cap A}$, $\mathbb{U} \setminus A$ is simply connected, contains the origin and has 1 on the boundary, the law of $\Phi_A(K)$ conditioned on $(K \cap A = \emptyset)$ is equal to law of K where Φ_A is the conformal map from $\mathbb{U} \setminus A$ onto $\mathbb{U}$ that preserves 1 and the origin (see Fig. 1).

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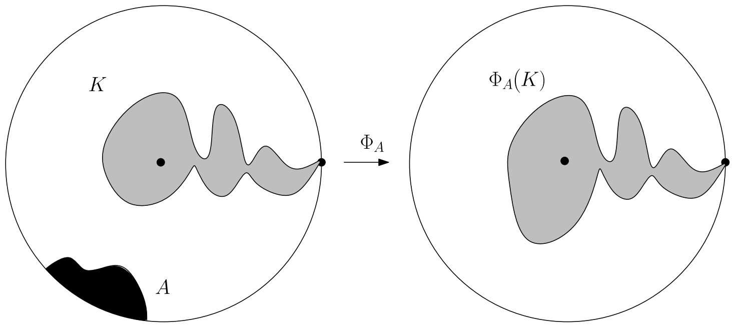

Fig. 1. Φ_A is the conformal map from $\mathbb{U} \setminus A$ onto $\mathbb{U}$ that preserves 0 and 1. Conditioned on $(K \cap A = \emptyset)$, $\Phi_A(K)$ has the same law as K.

The law of such a set K is called a radial restriction measure, by analogy with the chordal restriction measures defined in [6].

The main result of the present paper is the following classification and description of all radial restriction measures.

Theorem 1. 1. *(Characterization). A radial restriction measure is fully characterized by a pair of real numbers* (α, β) *such that*

$$\mathbb{P}\big[K \cap A = \emptyset\big] = |\Phi_A'(0)|^{\alpha}\, \Phi_A'(1)^{\beta}$$

where A is any closed subset of $\overline{\mathbb{U}}$ such that $A = \overline{\mathbb{U} \cap A}$, $\mathbb{U} \setminus A$ is simply connected, contains the origin and has 1 *on the boundary, and Φ_A is the conformal map from $\mathbb{U} \setminus A$ onto $\mathbb{U}$ that preserves* 0 *and* 1. *We denote the corresponding radial restriction measure by* $\mathbb{P}(\alpha, \beta)$.

2. *(Existence). The measure* $\mathbb{P}(\alpha, \beta)$ *exists if and only if*

$$\beta \geq \frac{5}{8}, \qquad \alpha \leq \xi(\beta) = \frac{1}{48}\left((\sqrt{24\beta + 1} - 1)^2 - 4\right).$$

We shall give an explicit construction of the measures $\mathbb{P}(\alpha, \beta)$ for all these admissible values of α and β. The function $\xi(\beta)$ is (as could be expected) the so-called disconnection exponent associated with the half-plane exponent β (see [7,3–5]).

It is worth observing that $|\Phi_A'(0)| \geq 1$ and that $\Phi_A'(1) \leq 1$. In Theorem 1, we see that the value of β is necessarily positive (and that therefore $\Phi_A'(1)^{\beta} \leq 1$), but the value of α can be negative or positive (as long as $\alpha \leq \xi(\beta)$), so that $|\Phi_A'(0)|^{\alpha}$ can be greater than one (but of course, the product $|\Phi_A'(0)|^{\alpha}\, \Phi_A'(1)^{\beta}$ cannot be greater than one which is guaranteed by the condition $\alpha \leq \xi(\beta)$).

This theorem is the counterpart of the classification of chordal restriction measures in [6] that we shall recall in the next section. It is worth noticing already that while the class of chordal conformal restriction measures was parametrized by a single parameter $\beta \geq 5/8$, the class of radial restriction samples is somewhat larger as it involves the additional parameter α. This can be rather easily explained by the fact that the radial restriction property is in a sense weaker than the chordal one. It involves an invariance property of the probability distribution under the action of the semi-group of conformal transformations that preserve both an inner point and a boundary point of the disc. In the chordal case, the semi-group of transformations were those maps that preserve two given boundary points (which is a larger family). Another way to see this is that the chordal restriction samples in the upper half-plane are scale-invariant, while the radial ones