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On non-standard limits of Brownian semi-stationary processes

Kerstin Gärtner[a], Mark Podolskij[b,*]

[a] *Institute of Population Genetics, University of Veterinary Medicine, Veterinärplatz 1, 1210 Vienna, Austria*
[b] *Department of Mathematics, University of Aarhus, Ny Munkegade 118, 8000 Aarhus C, Denmark*

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Abstract

In this paper we present some new asymptotic results for high frequency statistics of Brownian semi-stationary ($\mathcal{BSS}$) processes. More precisely, we will show that singularities in the weight function, which is one of the ingredients of a $\mathcal{BSS}$ process, may lead to non-standard limits of the realised quadratic variation. In this case the limiting process is a convex combination of shifted integrals of the intermittency function. Furthermore, we will demonstrate the corresponding stable central limit theorem. Finally, we apply the probabilistic theory to study the asymptotic properties of the realised ratio statistics, which estimates the smoothness parameter of a $\mathcal{BSS}$ process.

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1. Introduction

In the last years Brownian semi-stationary processes and their tempo-spatial extensions, *ambit fields*, have been widely studied in the literature. This class of models has been originally

* Corresponding author.
E-mail addresses: kerstin.gaertner@vetmeduni.ac.at (K. Gärtner), mpodolskij@creates.au.dk, m.podolskij@uni-heidelberg.de (M. Podolskij).

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proposed by Barndorff-Nielsen and Schmiegel [8] in the context of turbulence modelling. In their general form, Brownian semi-stationary processes without drift are defined as

$$X_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s W(ds), \quad t \in \mathbb{R}$$

where μ is a constant, W is a Brownian measure on $\mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ is a deterministic weight function with $g(t) = 0$ for $t \le 0$, and σ is a càdlàg processes. If σ is stationary and independent of W, then $(X_t)_{t\in\mathbb{R}}$ is stationary, which explains the name Brownian semi-stationary process. In the framework of turbulence modelling, $(X_t)_{t\in\mathbb{R}}$ denotes the velocity of a turbulent flow in the direction of the mean field measured at a fixed point in space. The stochastic process $(\sigma_t)_{t\in\mathbb{R}}$ embodies the *intermittency* of the dynamics of X. We refer to [8,10,9,11] for application of Brownian semi-stationary processes and ambit fields to turbulence modelling, and to [2,7] for further applications in mathematical finance and biology.

Recently, probabilistic properties of high frequency statistics of $\mathcal{BSS}$ processes have been investigated in several papers. We refer to a series of articles [4,5,13], which studies the asymptotic behaviour of (multi)power variation of $\mathcal{BSS}$ models. Typically, the weight function g considered in the aforementioned work has the form

$$g(x) = x^{\alpha} f(x), \quad \alpha \in (-1/2, 0) \cup (0, 1/2),$$

where f is a sufficiently smooth function slowly varying at 0 and with rapid decay at infinity. This type of weight functions satisfies $g \in \mathbb{L}^2(\mathbb{R})$, but $g' \notin \mathbb{L}^2(\mathbb{R})$ since g' is not square integrable near 0; in other words, the latter property means that 0 is the only *singularity point* of the weight function g. As a consequence, the process X is not a semimartingale. Moreover, its local behaviour corresponds to the one of a fractional Brownian motion with Hurst parameter $H = \alpha + 1/2$.

Understanding the limit theory for $\mathcal{BSS}$ processes requires an analysis of the following probability measure. For any $A \in \mathcal{B}(\mathbb{R})$, we define

$$\pi_n(A) := \frac{\int_A \{g(x+\Delta_n) - g(x)\}^2 dx}{\int_{\mathbb{R}} \{g(x+\Delta_n) - g(x)\}^2 dx}. \tag{1.1}$$

In the setting of weight functions as above, we deduce that $\pi_n \xrightarrow{d} \delta_0$ as $\Delta_n \to 0$, where δ_0 denotes the Dirac measure at 0 (cf. [4]). In this case the limit of the power variation of a $\mathcal{BSS}$ process is given as

$$\Delta_n \tau_n^{-p} \sum_{i=1}^{[t/\Delta_n]} |X_{i\Delta_n} - X_{(i-1)\Delta_n}|^p \overset{\text{u.c.p.}}{\Longrightarrow} m_p \int_0^t |\sigma_s|^p ds, \quad \text{as } \Delta_n \to 0, \tag{1.2}$$

where $m_p = \mathbb{E}[|\mathcal{N}(0,1)|^p]$, τ_n is a certain normalising sequence and $\overset{\text{u.c.p.}}{\Longrightarrow}$ stands for convergence in probability uniformly on compact sets. In [4,5] the asymptotic mixed normality of (multi)power variation is proved and the paper [13] studies the application of the limit theory to estimation of the *smoothness parameter* α. We remark that the asymptotic results are quite similar to the theory of power variations of continuous Itô semimartingales (cf. [6,15] among many others), although the methodologies of proofs are completely different.