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High order heat-type equations and random walks on the complex plane

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Abstract

A probabilistic construction for the solution of a general class of high order heat-type equations is constructed in terms of the scaling limit of random walks in the complex plane.

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1. Introduction

The connection between the solution of parabolic equations associated to second-order elliptic operators and the theory of stochastic processes is a largely studied topic [10]. The main instance is the *Feynman–Kac formula* (2), providing a representation of the solution of the heat equation (1) with possibly a potential $V \in C_0^\infty(\mathbb{R}^d)$

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = \dfrac{1}{2}\Delta u(t,x) - V(x)u(t,x), \quad t \in \mathbb{R}^+,\ x \in \mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

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in terms of an integral with respect to the measure of the Wiener process, the mathematical model of the Brownian motion [19]:

$$u(t,x)=\mathbb{E}^x\left[e^{-\int_0^t V(\omega(s))ds}u_0(\omega(t))\right]. \tag{2}$$

If the Laplacian in Eq. (1) is replaced by an higher order differential operator, i.e. if we consider for instance a Cauchy problem of the form

$$\begin{cases}\dfrac{\partial}{\partial t}u(t,x)=(-1)^{M+1}\Delta^M u(t,x)-V(x)u(t,x), & t\in\mathbb{R}^+,\ x\in\mathbb{R},\\ u(0,x)=u_0(x)\end{cases} \tag{3}$$

where $M>1$ is an integer, then a formula analog to (2), giving the solution of (3) in terms of the expectation with respect to the measure associated to a Markov process, is lacking. In fact, such a formula cannot be proven for semigroups whose generator does not satisfy the maximum principle, as in the case of Δ^M with $M>1$ [31].

One of the reasons making the higher powers of the Laplacian and Eq. (3) more difficult to handle than the traditional heat equation is the fact that, unlike the case where $M=1$, for $M>1$ the fundamental solution $G_t(x,y),\ t\in\mathbb{R}^+,\ x,y\in\mathbb{R}$, is not positive. In fact it has an oscillatory behavior, changing sign an infinite number of times [14]. Consequently it cannot be interpreted as the density of a positive probability measure, as the Gaussian transition densities of the Brownian motion, but only as the density of a signed measure. This fact has the troublesome consequence that if one uses G as a signed transition probability density in the construction of a generalized stochastic process with real path and independent increments, the resulting measure on $\mathbb{R}^{[0,+\infty)}$ would have infinite total variation. This fact was pointed out in [21] and can be regarded as a particular case of a general result by E. Thomas [32], generalizing Kolmogorov existence theorem to projective families of signed or complex measures instead of probability measures. In other words it is not possible to find a stochastic process X_t which plays for the parabolic equation (5) the same role that the Wiener process plays for the heat equation.

We would like to point out that the problem of the probabilistic representation of the solution of the Cauchy problem (3) presents some similarities with the problem of the mathematical definition of Feynman path integrals and the functional integral representation for the solution of the Schrödinger equation (see [27,18] for a discussion of this topic). Indeed in both cases it is not possible to implement an integration theory of Lebesgue type in terms of a bounded variation measure on a space of real paths [8]. This means that an analog of the Feynman–Kac formula for the parabolic equation (3), namely a representation for its solution of the form:

$$u(t,x)=\int_{\omega(0)=x}e^{-\int_0^t V(\omega(s))ds}u_0(\omega(t))d\mathbb{P}_M(\omega), \tag{4}$$

(where $\mathbb{P}_M$ should be some "measure" on a space of "paths" $\omega:[0,t]\to\mathbb{R}$) cannot be realized in terms of a well defined Lebesgue-type integral, but, in a weaker sense, in terms of a linear functional on a suitable class of "integrable functions", under some restrictions on the initial datum u_0 and the potential V. An analog approach has been successfully implemented in the case of the Schrödinger equation [1].

Several attempts have been made to relate such problem, as in the case $M=1$, to a random process, in particular for the case $M=2$ (known in the literature as the *biharmonic* operator).

One of the first approaches was introduced by Krylov [21] and continued by Hochberg [14], who introduced a stochastic pseudo-process whose transition probability function is not positive