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Stochastic Processes and their Applications 124 (2014) 1741–1772

stochastic processes and their applications

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The stochastic fluid–fluid model: A stochastic fluid model driven by an uncountable-state process, which is a stochastic fluid model itself[☆]

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Received 10 January 2012; received in revised form 11 December 2013; accepted 11 December 2013

Abstract

We introduce the Stochastic Fluid–Fluid Model, which offers powerful modeling ability for a wide range of real-life systems of significance. We first derive the infinitesimal generator, with respect to time, of the driving stochastic fluid model. We then use this to derive the infinitesimal generator of a particular Laplace–Stieltjes transform of the model, which is the foundation of our analysis. We develop expressions for the Laplace–Stieltjes transforms of various performance measures for the transient and limiting analysis of the model. This work is the first direct analysis of a stochastic fluid model that is Markovian on a continuous state space.

MSC: primary 60J25; secondary 60G17

Keywords: Stochastic fluid model; Markov chain; Laplace–Stieltjes transform; Transient analysis; Limiting distribution

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1. Introduction

Many important systems which provide essential services for society are complex and dynamic, and operate in an uncertain environment. Managers of such systems need to be able

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[☆] The authors would like to thank the Australian Research Council for funding this research through Discovery Project DP110101663.

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http://dx.doi.org/10.1016/j.spa.2013.12.006

to make properly informed decisions. Wrong decisions may have catastrophic effects on the performance of the system. An example of engineering importance here is the dimensioning and management of a router buffer in the Internet, which poses a complex and dynamic problem in a particularly uncertain environment. Sources of uncertainty for this problem include human behavior, the state of interacting parts of the Internet and the availability of shared resources. Designers of such systems need to be able to evaluate *performance measures* of their systems. These assist managers to make properly informed decisions. For this evaluation, appropriate stochastic models are necessary.

The uncertainty of such systems has been a major difficulty, as very complex, often large-scale systems, require more advanced models than the classical models that have been used in the past. Naturally, from the practical point of view, excessively complicated models are not desired either. Instead, flexible models which are parsimonious and computationally effective, are required.

Inspired by the engineering problems of the nature described above, primarily in high-speed telecommunications networks, a rapid development of novel, promising results in the area known as **stochastic fluid models** (SFMs) has been observed in recent years [1–6,10,12–16,28,29]. Stochastic fluid models are stochastic models with a state-space which can be thought of as two dimensional, consisting of a continuous *level* variable $X(t)$ and a finite *phase* variable $\varphi(t) \in \mathcal{S}$, where $\mathcal{S}$ is some finite set. The phase variable $\varphi(t)$ is often used to describe the state of the environment at time t, while the level variable $X(t)$ is used to describe some continuous aspect of the system at time t. Simple examples of two-phase processes are on/off mode of a source in a telecommunications buffer, peak/off-peak period in a telephone network, or wet/dry season in reservoir modeling. The level variable in these examples could be used to record the amount of data in the buffer, the number of customers in the network (which are both naturally discrete, but so large that we can think of them as a fluid), and the water level in the reservoir, respectively. In general, models with any (finite) number of phases are analyzed, and so the application potential extends far beyond the simplistic examples listed here. The model assumes that the transitions between phases occur according to the generator matrix $\mathbf{T}$ of some continuous-time Markov chain $\{\varphi(t), t \geq 0\}$. Furthermore, this underlying Markov chain affects the level variable $X(t)$ in the following way. At time t, when $\{\varphi(t), t \geq 0\}$ is in some phase $i \in \mathcal{S}$, the rate of increase of the fluid level is given by the constant c_i, which may be positive, negative or zero. For example, in reservoir modeling we could assume that when the phase is dry, the fluid level in the reservoir is decreasing at some rate, due to the consumption of the water. Alternatively, when the phase is wet, the fluid level in the reservoir is increasing. We say that the underlying Markov chain is what drives the fluid level $X(t)$ at time t. A SFM is hence a two-dimensional Markov process, $\{(\varphi(t), X(t)), t \geq 0\}$.

SFMs have been highly successful in modeling the behavior of telecommunication networks. However, it has quickly become evident that SFMs have tremendous application potential in many other areas, well beyond telecommunications. These areas encompass all areas of industry, including insurance and manufacturing/management systems, as well as environmental problems, such as coral modeling and hydro-power management, as examples [7,12,21].

In this paper, we are interested in the following modeling potential. Suppose that, besides the level $X(t)$ at time t, we would like to model some other continuous performance measure $Y(t)$, such as the net profit at time t for example. To illustrate this, consider a hydro-power generator, which can be operated "on design" and "off design". The latter causes increased wear, and is less efficient, but can be useful to optimize overall system operation. The generator must be periodically maintained, in order to improve its performance and prolong its lifespan. An important related problem is the evaluation of maintenance strategies and the impact of