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Representation of Gaussian isotropic spin random fields

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Abstract

We develop a technique for the construction of random fields on algebraic structures. We deal with two general situations: random fields on homogeneous spaces of a compact group and in the spin line bundles of the 2-sphere. In particular, every complex Gaussian isotropic spin random field can be represented in this way. Our construction extends P. Lévy's original idea for the spherical Brownian motion.

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1. Introduction

In recent years the investigation of random fields on algebraic structures has received a renewed interest from a theoretical point of view [17], but mainly motivated by applications concerning the modeling of the Cosmic Microwave Background data (see [1,7,13] e.g. and also the book [14]). Actually one of the features of this radiation, the *temperature*, is well modeled as a single realization of a random field on the sphere $\mathbb{S}^2$.

Moreover thanks to the ESA satellite Planck mission, new data concerning the *polarization* of the CMB will soon be available and the modeling of this quantity has led quite naturally to the investigation of spin random fields on $\mathbb{S}^2$, a subject that has already received much attention (see [8,10,12] and again [14, Chapter 12] e.g.).

The object of this paper is the investigation of Gaussian isotropic random fields or, more precisely random sections, in the homogeneous line bundles of $\mathbb{S}^2$. In this direction we first investigate the simpler situation of random fields on the homogeneous space $\mathscr{X} = G/K$ of a

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compact group G. Starting from P. Lévy's construction of his "mouvement brownien fonction d'un point de la sphère de Riemann" in [11], we prove that to every square integrable bi-K-invariant function $f : G \to \mathbb{R}$ a real Gaussian isotropic random field on $\mathscr{X}$ can be associated and also that every real Gaussian isotropic random field on $\mathscr{X}$ can be obtained in this way.

Then, turning to our main object, we prove first that every random section of a homogeneous vector bundle on $\mathscr{X}$ is actually "equivalent" to a random field on the group G, enjoying a specific pathwise invariance property. This fact is of great importance as it reduces the investigation of random sections of the homogeneous vector bundle to a much simpler situation and is our key tool for the results that follow.

We are then able to give a method for constructing Gaussian isotropic random sections of the homogeneous line bundles of $\mathbb{S}^2$ and prove that every complex Gaussian isotropic random section can actually be obtained in this way. More precisely given $s \in \mathbb{Z}$, we prove that to every function $f : SO(3) \to \mathbb{C}$ which is bi-s-associated, i.e. that transforms under both the right and left action of the isotropy group $K \simeq SO(2)$ according to its s-th linear character, one can associate a Gaussian isotropic random section of the s-homogeneous line bundle and also that every complex Gaussian isotropic random section is represented in this way. In some sense this extends the representation result for Gaussian isotropic random fields on homogeneous spaces described above: a bi-K-invariant function being associated to the 0-th character and a random field on $\mathbb{S}^2$ being also a random section of the 0-homogeneous line bundle.

In [11] P. Lévy proves the existence on the spheres $\mathbb{S}^m$, $m \geq 1$, of a Gaussian random field T such that $T_x - T_y$ is normally distributed with variance $d(x, y)$, d denoting the distance on $\mathbb{S}^m$. The existence of such a random field on a more general Riemannian manifold has been the object of a certain number of papers ([2,6,19] e.g.). We have given some thought about what a P. Lévy random section should be. We think that our treatment should be very useful in this direction but we have to leave this as an open question.

The plan is as follows. In Section 2 we recall general facts concerning Fourier expansions on the homogeneous space $\mathscr{X}$ of a compact group G. In Section 3 we introduce the topic of isotropic random fields on $\mathscr{X}$ and their relationship with positive definite functions enjoying certain invariance properties. These facts are the basis for the representation results for real Gaussian isotropic random fields on $\mathscr{X}$ which are obtained in Section 4.

In Section 5 we introduce the subject of random sections of homogeneous vector bundles on $\mathscr{X}$ and in particular in Section 6 we consider the case $\mathscr{X} = \mathbb{S}^2$. We were much inspired here by the approach of [12], but the introduction of pullback random fields considerably simplifies the understanding of the notions that are involved.

In Section 7 we extend the construction of Section 4 as described above and prove the representation result for complex Gaussian isotropic random sections of the homogeneous line bundles of $\mathbb{S}^2$. When dealing with such random sections different approaches are available in the literature. They are equivalent but we did not find a formal proof of this equivalence. So Section 8 is devoted to the comparison with other constructions [8,10,12,16]. In particular we show that the construction of spin random fields on $\mathbb{S}^2$ of [8] is actually equivalent to ours, the difference being that their point of view is based on an accurate description of local charts and transition functions, instead of our more global perspective.

2. Fourier expansions

Throughout this paper $\mathscr{X}$ denotes the (compact) homogeneous space of a topological *compact* group G. Therefore G acts transitively on $\mathscr{X}$ with an action $x \mapsto gx$, $g \in G$. $\mathscr{B}(\mathscr{X})$, $\mathscr{B}(G)$