Available online at www.sciencedirect.com

SciVerse ScienceDirect

Stochastic Processes and their Applications 123 (2013) 651–674

stochastic processes and their applications

www.elsevier.com/locate/spa

Convergence in total variation on Wiener chaos

Ivan Nourdin*, Guillaume Poly

Université de Lorraine and Université Paris Est, France

Received 12 May 2012; received in revised form 3 October 2012; accepted 5 October 2012
Available online 13 October 2012

Abstract

Let $\{F_n\}$ be a sequence of random variables belonging to a finite sum of Wiener chaoses. Assume further that it converges in distribution towards F_∞ satisfying $\mathrm{Var}(F_\infty) > 0$. Our first result is a sequential version of a theorem by Shigekawa (1980) [23]. More precisely, we prove, without additional assumptions, that the sequence $\{F_n\}$ actually converges in total variation and that the law of F_∞ is absolutely continuous. We give an application to discrete non-Gaussian chaoses. In a second part, we assume that each F_n has more specifically the form of a multiple Wiener–Itô integral (of a fixed order) and that it converges in $L^2(\Omega)$ towards F_∞. We then give an upper bound for the distance in total variation between the laws of F_n and F_∞. As such, we recover an inequality due to Davydov and Martynova (1987) [5]; our rate is weaker compared to Davydov and Martynova (1987) [5] (by a power of 1/2), but the advantage is that our proof is not only sketched as in Davydov and Martynova (1987) [5]. Finally, in a third part we show that the convergence in the celebrated Peccati–Tudor theorem actually holds in the total variation topology.

MSC: 60F05; 60G15; 60H05; 60H07

Keywords: Convergence in distribution; Convergence in total variation; Malliavin calculus; Multiple Wiener–Itô integral; Wiener chaos

1. Introduction

In a seminal paper of 2005, Nualart and Peccati [19] discovered the surprising fact that convergence in distribution for sequences of multiple Wiener–Itô integrals to the Gaussian is equivalent to convergence of just the fourth moment. A new line of research was born. Indeed, since the publication of this important paper, many improvements and developments on this

* Corresponding author.
E-mail addresses: inourdin@gmail.com (I. Nourdin), guillaume.poly@crans.org (G. Poly).

0304-4149/$ - see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.spa.2012.10.004

theme have been considered. (For an overview of the existing literature, we refer the reader to the book [14], to the survey [12] or to the constantly updated web page [13].)

Let us only state one of these results, whose proof relies on the combination of Malliavin calculus and Stein's method (see, e.g., [14, Theorem 5.2.6]). When F, G are random variables, we write $d_{TV}(F, G)$ to indicate the total variation distance between the laws of F and G, that is,

$$d_{TV}(F, G) = \sup_{A \in \mathcal{B}(\mathbb{R})} |P(F \in A) - P(G \in A)| = \frac{1}{2} \sup_{\phi} |E[\phi(F)] - E[\phi(G)]| ,$$

where the first (resp. second) supremum is taken[1] over Borel sets A of $\mathbb{R}$ (resp. over continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ which are bounded by 1).

Theorem 1.1. *If $k \geqslant 2$ is an integer, if F is an element of the kth Wiener chaos $\mathcal{H}_k$ satisfying $E[F^2] = 1$ and if $N \sim \mathcal{N}(0, 1)$, then*

$$d_{TV}(F, N) \leqslant \sqrt{\frac{4k-4}{3k}} \sqrt{\left|E\left[F^4\right] - 3\right|}.$$

As an almost immediate corollary of Theorem 1.1, we get the surprising fact that if a sequence of multiple Wiener–Itô integrals with unit variance converges in distribution to the standard Gaussian law, then it automatically converges in total variation [14, Corollary 5.2.8]. The main thread of the present paper is the seek for other instances where such a phenomenon could occur. In particular, a pivotal role will be played by the sequences having the form of a (vector of) multiple Wiener–Itô integral(s) or, more generally, belonging to a *finite* sum of Wiener chaoses. As we said, the proof of Theorem 1.1 relies in a crucial way to the use of Stein's method. In a non-discrete framework (which is the case here), it is fairly understood that this method can give good results with respect to the total variation distance only in dimension one (see [3]) and when the target law is Gaussian (see [4]). Therefore, to reach our goal we need to introduce completely new ideas with respect to the existing literature. As anticipated, we will manage to exhibit three different situations where the convergence in distribution turns out to be equivalent to the convergence in total variation. In our new approach, an important role is played by the fact that the Wiener chaoses enjoy many nice properties, such as hypercontractivity (Theorem 2.1), product formula (2.7) or Hermite polynomial representation of multiple integrals (2.3).

Let us now describe our main results in more detail. Our first example focuses on sequences belonging to a finite sum of chaoses and may be seen as a sequential version of a theorem by Shigekawa [23]. More specifically, let $\{F_n\}$ be a sequence in $\bigoplus_{k=0}^{p} \mathcal{H}_k$ (where $\mathcal{H}_k$ stands for the kth Wiener chaos; by convention $\mathcal{H}_0 = \mathbb{R}$), and assume that it converges in distribution towards a random variable F_∞. Assume moreover that the variance of F_∞ is not zero. Let d_{FM} denote the Fortet–Mourier distance, defined by

$$d_{FM}(F, G) = \sup_{\phi} |E[\phi(F)] - E[\phi(G)]| ,$$

where the supremum is taken over 1-Lipschitz functions $\phi : \mathbb{R} \to \mathbb{R}$ which are bounded by 1. We prove that there exists a constant $c > 0$ such that, for any $n \geqslant 1$,

$$d_{TV}(F_n, F_\infty) \leqslant c \, d_{FM}(F_n, F_\infty)^{\frac{1}{2p+1}} . \tag{1.1}$$

[1] One can actually restrict to *bounded* Borel sets without changing the value of the supremum; this easy remark is going to be used many times in the forthcoming proofs.